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## CIII. *On the Variation of the Resistance of Thermionic Valves at High Frequencies.*

*By* S. K. MITRA, *D.Sc., Khaira Professor of Physics, and* BHABANI CHARAN SIL, *M.Sc., Khaira Research Scholar in Physics, Calcutta University* *.

### 1. INTRODUCTION.

THE influence of electrons and ions in changing the dielectric constant and conductivity of a gas has of late received a good deal of attention on account of its bearing on the mode of propagation of radio waves in the ionized upper atmosphere †. The nature of the influence can be most easily understood by considering the behaviour of an electron situated between the plates of a unit condenser in vacuum ‡. An alternating voltage applied to the plates sets the electron in motion, and the current produced thereby lags a quarter period behind the applied voltage. The effect of such an electron in motion can be shown to be equivalent to that of a self inductance of value $L=\frac{m}{e^2}$, shunted across the unit condenser. This effective self

* Communicated by the Authors.

† Eccles, Proc. Roy. Soc. A, lxxxvii. p. 79 (1912); Salpeter, *Phys. Zeit.* xiv. s. 20 (1913); Larmor, Phil. Mag. (6) xlviii. p. 1025 (1924) Elias, *Jahrb. f. drahtl. Tel.* Band xxvii. (1926); Lassen, *Jahrb. f. drahtl. Tel.* Band xxviii. (Nov. 1926).

‡ P. O. Pedersen, 'Propagation of Radio Waves,' published by Denmarks Naturvidenskabelige Samfund, p. 87 (1927).

inductance in its turn may be regarded as equivalent to a reduction in the dielectric constant of the medium by an amount $\Delta\xi = \frac{4\pi e^2}{\omega^2 m}$ ($e$, $\omega$, $m$ having their usual significance). It should be noted that since the electron is moving in free space devoid of matter, it does not on the whole cause any dissipation of energy. During part of the cycle it absorbs energy from the field and is set in motion, and during another part it gives energy back to the field and is brought to rest. The resistance of the medium thus remains infinite, as it had been previous to the introduction of the electron. If, instead of a single electron, there are N electrons, the expression for the dielectric constant is correspondingly modified. It is interesting to note that even in the case of a swarm of N electrons there is no loss * of electrical energy due to collision among the electrons themselves, and consequently the resistance of the medium might still be regarded as infinite.

Conditions are changed, however, if the electrons move in a gas atmosphere and collide with the molecules. In this case the molecules of the gas-filled space absorb energy from the electrons and the resistance of the space is brought down from infinity to some finite value. The expressions for the dielectric constant and the conductivity in this case are modified as follows †:

$$\Delta\xi = -\mathrm{N}\frac{4\pi e^2}{m}\cdot\frac{1}{\omega^2+v^2} = -\mathrm{K}\ (\text{say}). \quad . \quad . \quad (1)$$

$$\sigma = -\frac{v}{4\pi c^2}\cdot \mathrm{K}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $v$ is the number of collisions per second between electrons and gas molecules.

During recent years a large number of investigations have been carried out to measure the dielectric constant at various frequencies, either of an ionized gas in which both electrons and gas molecules are present ‡, or of a pure

* Pedersen, Proc. Inst. Rad. Eng. xvii. p. 1762 (Oct. 1929).

† Pedersen, 'Propagation of Radio Waves,' p. 91. See also Benner, *Annalen der Phys.* (3) vii. p. 993 (1929).

‡ Bergmann & Düring, *Ann. der Phys.* ser. v. vol. i. no. 48 (May 1929); Appleton & Childs, Phil. Mag. (10) lxvii. p. 969 (Dec. 1930); Jonescu & Mihul, *Compt. Rendus*, cxci. (Dec. 1930). See, however, H. Gurton & J. Clement, *Compt. Rendus*, clxxxiv. (Feb. 1927); criticism thereto by Pedersen, *loc. cit.* p. 93, Bergmann & Düring, *loc. cit.*, and reply by H. Gutton, *L'onde Electrique*, vii. (Jan. 1928) and *Annales de Physique*, xiii. (Jan. 1930).

electronic atmosphere as may be found in a highly evacuated triode value *. Results show that a change in the dielectric constant is produced in the same sense as indicated by the above theory when extraneous causes are eliminated. Comparatively little work seems to have been done on the measurement of the conductivity, especially its variation with frequency. The purpose of the present paper is to investigate, both theoretically and experimentally, the change produced in the conductivity of a space containing free electrons and bounded by two electrodes (as is usually found in a thermionic valve) when a high-frequency E.M.F. is applied across the electrodes.

From what has been said before it might appear that in the high vacuum of a triode valve, where the collisions between electrons and the remnant gas molecules are few and far between, the only effect of high frequency will be to produce an oscillatory motion of the electrons resulting in a slight reduction in the capacity due to the inertial effect. Since there is no collision there is no dissipation of energy, and, consequently, there is no change of resistance. But it must not be lost sight of, that though the electrons do not collide with any gas molecules, they do collide with the electrodes where they give up both their kinetic energy and their charges. In this respect the resistance of a triode valve differs from an ordinary resistance. When a current flows through an ordinary resistive medium the dissipation of energy occurs in the medium itself, the electrodes merely acting as current leads ; but in a thermionic valve, on account of the scarcity of gas molecules, the dissipation of energy occurs not in the interelectrode space but rather in the electrodes themselves, which not only lead the current in and out, but also dissipate the kinetic energy of the electrons converting it to heat. In fact, the dissipation of energy depends entirely upon the number of electrons reaching the plate and on their kinetic energy, the latter being independent of the paths by which the electrons might have travelled up to the plate.

We will show in the present paper that if we assume the electrons in the interelectrode space to be endowed with

* Benham, Phil. Mag. (7) xi. p. 457 (Feb. 1931, Suppl.) ; Hartshorn (Proc. Phys. Soc. xli. p. 113) seems to have obtained a variation of interelectrode capacity in the opposite sense to that of the Eccles-Larmor theory. This result, as well as a similar one obtained by Ballantine (Journ. Frank. Inst. ccvi. no. 2, p. 159 (Aug. 1928)), can be explained as due to neglecting the inertial effect of the electrons. *Cf.* Benham, *loc. cit.* p. 473 ; also Wireless Eng. and Exp. Wireless, viii. p. 489 (Sept. 1931).

a Maxwellian distribution of velocity, then the number of electrons reaching the plate per unit time under the action of an applied E.M.F. will depend not only on the magnitude of the E.M.F. but also on its duration, specially when it is so small as to be comparable with the time of flight of the electrons from one electrode to another. As the duration of the pulse is diminished, or in other words, as its frequency is increased, the number of electrons reaching per unit time will diminish and the resistance will increase *.

That the number of electrons striking the plate will depend upon the frequency is also seen from the fact that the amplitude of the electron moving under the action of an alternating field is given by $\frac{e\mathrm{F}_m}{\omega^2 m}$ ($\mathrm{F}_m$ is the peak value of the applied alternating field). As the frequency is raised the amplitude is diminished, and the electrons have less chance of reaching the plates, which form as it were the boundary of the space containing the electrons. It will also be shown that this hypothesis is fully borne out by experimental result. It should be noted that the sense of the change of resistance with frequency is the same as that given by equation (2) for the case of an ionized gas. A rise in frequency is accompanied by a rise in resistance. This must be so because the mechanism of the loss of energy is the same in both cases. In the one the gas molecules on either side of the oscillating electron form the boundary, and in the other these are replaced by the walls of the electrodes which restrict the amplitude of the electrons.

## 2. Theory of the Variation of Resistance with Frequency.

Consider an electron situated midway between the plates of a parallel plate condenser. Under the action of the high-frequency E.M.F. the electron will oscillate about its mean position provided the peak value of the high-frequency E.M.F. is not sufficiently high to drive it to one of the

* In the paper already referred to, and also in Expt. Wireless, viii. pp. 415–16 (Aug. 1931), Hartshorn shows that the resistance of the thermionic valve at very high frequencies should be *smaller* than that at very low frequencies by about 11 per cent. The results obtained by us show that at very high frequencies the resistance *increases* with the rise of frequency. The considerations on which Hartshorn's results are based are altogether different from ours. Moreover, we have found an *increase* in resistance of more than 100 per cent. in going from $10^7$ to $6\times10^7$ cycles, as against Hartshorn's *decrease* of only 11 per cent. in going from very *low* to very high frequencies.

condenser plates. As mentioned in the introduction, the presence of this oscillating electron will, on the average, cause no dissipation of energy in the condenser. During one part of the cycle it will absorb energy and will be set in motion, and during another part this energy will be given back to the electric field between the plates.

Let us now consider what will happen if the electron is driven to one of the plates. The electron gives up its charge as well as its kinetic energy to the plate. The former gives us the conduction current and the latter is spent in heating the condenser plates. We are now justified in conceiving the space between the plates as having a finite conductivity or resistance, and the latter is measured in the usual way by dividing the applied voltage by the quantity of electricity carried to the plate per unit time under the action of the voltage.

Now suppose that the space between the condenser plates contains a source of electrons which replenishes the electrons lost by absorption in the plate. Let us further suppose that the electrons are endowed with a Maxwellian distribution of velocity.

If an alternating voltage is applied between the two plates of the condenser, the electrons coming out of the source will oscillate within the space between the plates. When one of the plates becomes positive during a half cycle, the electrons which were moving towards that plate are caught by the electric field and carried to the plate. If the frequency of alternation is not high, some of those which were at rest, and even some of those which were moving in the opposite direction to the field, will also reach the plate and give up their charges to it.

Now let $t$ be the time during which the positive half of the impressed high-frequency E.M.F. (supposed to be constant during the interval) acts, and let $n$ be the number of electrons reaching the plate during this time. Then the conductivity is evidently given by $\frac{ne}{tV_a}$, where $V_a$ is the potential difference between the plates. A little consideration will show that the quantity $\frac{n}{t}$ is not constant, as is usually supposed, but that it depends on $t$. This is particularly so when the time-period $2t$ of the high frequency impressed E.M.F. becomes comparable with the time taken by the electron to reach the plate. The following will make the point clear :—

Take the case of an electron at a distance $s$ from one of

the condenser plates and moving with a velocity $v_0$ along $s$ towards it. Let the velocity $v_0$ be such that the electron, if left to itself, will fail to reach the plate during the time $t$. If, however, a field is applied between the plates, then it is possible that, under the influence of this field, the electron will be able to strike the plate and give up its charge to the latter. It is, therefore, possible to find a critical velocity $v_0$, with which the electron must be moving initially in order that the applied field of frequency $\frac{1}{2t}$ is just able to carry the electron to the plate at the end of the interval $t$. It is evident that all the electrons which have been moving with a velocity greater than this critical velocity $v_0$ at the instant when the field began to act will also reach the plate within the time $t$. The number of electrons reaching the plate will thus depend not only on the time $t$ but also on the total number of electrons lying within the velocity range $v_0$ to $\infty$. In fact, this latter is a measure of the conductivity of the space between the condenser plates for the particular frequency to which the critical velocity $v_0$ corresponds.

The problem of finding out the conductivity for a particular frequency $f = \frac{1}{\mathrm{T}}$ (where T is the time period of alternation) is thus reduced to finding out ($a$) the critical initial velocity $v_0$, such that an electron moving initially with this velocity is just able to reach the plate at the end of the period $\frac{\mathrm{T}}{2}$, during which the impressed E.M F. acts in one direction, and ($b$) calculating the number $n$ of electrons moving in the velocity range $v_0$ to $\infty$. The conductivity for the particular frequency $\frac{1}{\mathrm{T}}$ is then proportional to $n$.

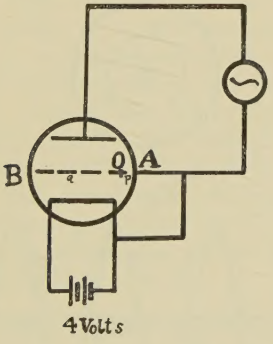
### 3. Calculation of the Change of Resistance with Frequency.

As it was inconvenient to obtain a parallel plate condenser with a source of electrons between its plates, the condenser actually employed for the measurement was a cylindrical one formed by the grid and the plate of a triode valve. The source of the electrons, namely, the filament being along the axis of the cylindrical condenser was outside the condenser space. But the electrons emitted from the filament had sufficient thermic velocity to reach the plate-grid condenser space through the mesh of the grid. The conductivity

of this space as modified by the presence of the electrons was measured for various frequencies by the distuning method to be described later. The grid was normally connected to the negative of the heating battery, and the high-frequency E.M.F. for measuring the resistance was applied between the grid and the plate (fig. 1). We will therefore proceed to calculate, on the basis of the theory outlined in the preceding section, the variation of the resistance of this space with frequency due to the presence of electrons with a Maxwellian velocity distribution.

It will be observed that in order to reach the plate the electrons have to move in two distinct fields of force : (*a*) the field between the filament and the grid-cylinder; and (*b*) that

Fig. 1.




between the grid-cylinder and the plate when the impressed E.M.F. was acting. By "grid-cylinder" we mean the imaginary cylindrical surface on which the grid wire is coiled in the form of a helix. The potential of the grid-cylinder is not in general the same as the potential of the grid wire. Its potential $V_0$ is a function of $\mu$, the amplification factor of the valve, of $V_a$ the anode potential, and of $V_g$ the grid potential, and is given by the formula

$$V_0 = \frac{\mu V_g + V_a}{\mu + 1}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

3 (1). *Some Simplifying Assumptions.*

In order to carry out the above calculations and arrive at a formula giving the proportionate number of electrons which will reach the plate corresponding to a particular

time-interval $t$, we will make a number of simplifying assumptions as described below :—

(*a*) The high-frequency voltage used in the measurement varied sinusoidally, as shown by the full line curve, fig. 3. We will, however, assume that the time-voltage curve is as shown by the dotted line. In other words, we assume that in the positive half-cycle the voltage suddenly rises from zero value to the value $\frac{2V_m}{\pi}$ at $t=0$, T, 2T, etc., maintains that value during the intervals $t=0$ to $t=\frac{T}{2}$, $t=T$ to $t=\frac{3T}{2}$, etc., and then suddenly drops to zero value again at $t=\frac{T}{2}$, $\frac{3T}{2}$, etc.

Fig. 2.

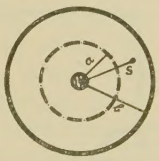
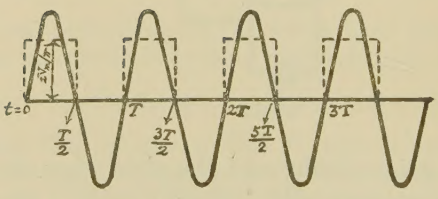


Fig. 3.




$V_m$ being the peak value of the high-frequency voltage, $\frac{2V_m}{\pi}$ is obviously the mean value during one half-cycle.

We are not concerned with what happens during the negative half-cycle, because no current flows during this period and nothing is contributed towards the conductivity.

In the actual experiment the value of $\frac{2V_m}{\pi}$ was ·21 volt.

(*b*) In calculating the time taken by the electron to reach the plate under the influence of the impressed plate-grid voltage, we will assume that the source of electrons is the grid-cylinder and that the electrons reaching the plate are all drawn from this region. Thus if $V_x$ is the potential at

a point distant $x$ from the end A of the grid-cylinder (fig. 1) then the concentration in the neighbourhood of $x$ is in accordance with Boltzmann's law $N\epsilon^{-\frac{eV_x}{KT}}$, where N is the electronic concentration near A. Thus the contribution to the plate current from any portion of the cylinder varies exponentially with the potential at that point.

(*c*) Since there is a drop of 4 volts along the filament due to its resistance, it follows that the voltage of the grid-wire is not constant throughout its length with respect to the portion of the filament immediately in front of it, *i. e.*, the portion of the filament near the foot of the perpendicular dropped from the point under consideration (of the grid-wire) on to the filament. The voltage varies between zero at A to −4 volt at B (fig. 1). The voltage along the length of the plate also varies in a like manner. During the positive half-cycle it varies between ·21 at A to −4+·21 at B, and during the negative half-cycle between −·21 at A to −(·21+4) at B. Due to the joint influence of the plate voltage and the grid voltage, the voltage along the grid-cylinder varies between ·035 volt at A to −3·3 volt at B during the positive half-cycle, and between −·035 at A to −3·37 at B during the negative half-cycle (see equation (3)). The mean voltage drop along the grid-cylinder thus varies between 0 at A to −3·335 volt at B.

(*d*) During the positive half-cycle the voltage of the grid-cylinder has a value ·035 volt at A and −3·3 volt at B, the variation along the length being linear. It is easy to calculate that at the point O (fig. 1), distance ·02 cm. from A, the voltage of the grid-cylinder with respect to the portion of the filament "in front of it" is zero. (The total length of the grid-cylinder is 2 cm.) We divide the grid-cylinder into two portions, P and Q. Portion P from the end A to O is taken to be approximately at the same potential as that of the point A, namely, ·035 volt, and portion Q from the point O to the end B is everywhere lower than the potential of the filament, being zero at O and −3·3 volt at B. On the assumption (*b*) that the electrons drawn to the plate all come from the neighbourhood of the grid-cylinder and that the contribution from any portion is proportional to the concentration there, we can find out the comparative contributions to the plate current by the two portions P and Q of the grid-cylinder. The contribution by the portion P is proportional to ·02N, since N is the concentration near A

and it has been assumed to be constant over the small length ·02 cm., A to O. The contribution by the portion Q is given by $N\int_0^{1\cdot98} \epsilon^{-\frac{eV_x}{KT}} dx$, where $V_x$ is the potential at a distance $x$ from O. The expression when evaluated gives for the value of the contribution by the portion Q as proportional to ·021 N approximately. It is thus seen that although the portion P is only one-hundredth part in length of the portion Q, yet the contribution from the former is nearly the same as that from the latter.

(*e*) The electrons which reach the anode might be divided into three groups in accordance with the nature of the paths they follow :—

(i.) Electrons which start from the grid-cylinder and move outward towards the anode in a radial direction.

(ii.) Electrons which start from the grid-cylinder and move inward towards the filament in a radial direction. Of these, those which start from the region Q of the grid-cylinder move in an accelerating field and have no chance of returning back to the grid-cylinder. But those which start inward from region P are moving in a retarding field and consequently may turn back if the retarding field produced during the positive half-cycle of the impressed E.M.F. is sufficiently strong. Such electrons thus repass through the grid-cylinder and reach the plate by the action of the impressed field between the grid and the anode.

(iii ) The electrons which move inwards but, instead of travelling in a radial direction, leave the grid-cylinder obliquely, get past the filament, and return after describing an orbit round the filament.

3 (2). *Calculation of the Time of Flight of the Electrons.*

We will now proceed to calculate the time taken by electrons of classes (i.) and (ii.) to reach the anode for various initial velocities. It has not been found possible to calculate with any degree of approximation the time of passage for class (iii.), and the contribution to the conductivity of this group of electrons has been neglected. To this reason might be attributed the discrepancy between the calculated and the observed values of the conductivity which has been found for the case of large time-intervals (low frequencies).

(a) *Time of flight for an electron of class* (i.) *from grid to anode.*—Let the electron be at a distance S from the centre of the filament (fig. 2). If an electric field is applied between the grid and the plate, the plate being at a higher potential, the force on the electron will vary inversely as its distance S from the centre of the filament. Since the force is in a direction away from the filament, the equation of motion of the electron may be written as

$$\frac{d^2S}{dt^2} = \frac{K}{S}.$$

If $F_m$ is the peak value of the impressed high-frequency oscillations between grid and plate, then, in accordance with assumption (3·1 *a*)

$$K = \frac{2F_m}{\pi m}.$$

Integrating the equation, we have

$$\frac{1}{2}\left(\frac{dS}{dt}\right)^2 = K \log S + C.$$

Let the initial velocity of the electron be $v_0$, *i. e.*,

$$\frac{dS}{dt} = v_0, \text{ when } S = a.$$

Then

$$C = \tfrac{1}{2}v_0^2 - K \log a,$$
$$= K \log P, \text{ say}$$

where

$$P = \epsilon^{\left(\frac{1}{2K}v_0^2 - \log a\right)}.$$

Thus

$$\frac{dS}{dt} = \sqrt{2K \log PS}.$$

$$\therefore \quad t = \frac{1}{\sqrt{2K}} \int_a^b \frac{dS}{\sqrt{\log PS}}.$$

Put $\log PS = y$.

Then

$$t = \frac{1}{\sqrt{2K}\,\epsilon^{\left(\frac{1}{2K}v_0^2 - \log a\right)}} \int_{\frac{1}{2K}v_0^2}^{\frac{1}{2K}v_0^2 + \log\frac{b}{a}} y^{-\frac{1}{2}} \epsilon^y \, dy$$

$$= \frac{1}{\sqrt{2K}\epsilon^{\left(\frac{1}{2K}v_0^2 - \log a\right)}} \left[ \int_0^{\frac{1}{2K}v_0^2 + \log\frac{b}{a}} y^{-\frac{1}{2}}\epsilon^y\, dy - \int_0^{\frac{1}{2K}v_0^2} y^{-\frac{1}{2}}\epsilon^y\, dy \right].$$

The integrals are of the form $\int x^{-\frac{1}{2}}\epsilon^x\, dx$, and for obtaining numerical results the following series expansion may be used :

$$\int x^{-\frac{1}{2}}\epsilon^x\, dx = \frac{\epsilon^x}{x^{\frac{1}{2}}}\left[1 + \frac{1}{2x} + \frac{3}{(2x)^2} + \frac{3\,.\,5}{(2x)^3} + \ldots\ldots\right],$$

also

$$= \frac{\epsilon^x}{x^{\frac{1}{2}}}\left[2x - \frac{(2x)^2}{3} + \frac{(2x)^3}{3\,.\,5} - \ldots\ldots\right].$$

From these equations the values of $t$ are found for different values of $v_0$.

(b) *Time of flight of the electrons of class* (ii.), *grid to filament and back to anode.*—As mentioned previously, electrons moving inward radially from the grid-cylinder towards the filament may also reach the plate under the influence of the positive field developed between the grid-cylinder and the filament due to the impressed E.M.F. between the former and the plate. The process is as follows :—An electron starting inward with velocity $v_0$ from the region P (see assumption 3·1 $d$) of the grid-cylinder is caught up in a retarding field between the grid-cylinder and the filament, and its velocity is reduced to zero in time $t_1$ after it has travelled a certain distance "$d$." It then moves back towards the grid-cylinder and reaches the same after another lapse of time $t_1$. Thenceforward it moves in the accelerating field between the grid-cylinder and the plate and reaches the latter after a further lapse of time $t_2$. The total time taken by the electron to reach the plate is thus equal to $t = t_1 + t_1 + t_2$. $t_2$ can be calculated as in the previous case. In order to calculate $t_1$ the equation in case ($a$) has to be slightly modified. As before, we have

$$\frac{1}{2}\left(\frac{d\mathrm{S}}{dt}\right)^2 = \mathrm{K}\log\mathrm{S} + \mathrm{C}.$$

Let the velocity be reduced to zero when the electron has traversed inward a distance $d$ in time $t_1$.

Then

$$\frac{d\mathrm{S}}{dt} = 0, \text{ when } \mathrm{S} = d.$$

Thus $$C = -K \log d,$$

and $$\frac{1}{2}\left(\frac{dS}{dt}\right)^2 = K \log \frac{S}{d}.$$

When $S=a$, *i. e.*, when the electron reaches the grid-cylinder its velocity $v_0$ is given by

$$\tfrac{1}{2}v_0^2 = K \log \frac{a}{d},$$

$v_0$ is obviously also the initial negative velocity of the electron at the moment when the field is applied.

To find the time $t_1$ taken by the electron to move from $S=d$ to $S=a$, we have

$$\frac{dS}{dt} = \sqrt{2K \log \frac{S}{d}}.$$

$$\therefore \qquad t_1 = \frac{1}{\sqrt{2K}} \int_d^a \frac{dS}{\sqrt{\log \frac{S}{d}}}.$$

Changing the variable by putting

$$y = \log \frac{S}{d},$$

we have $$t_1 = \frac{d}{\sqrt{2K}} \int_0^{\log \frac{a}{d}} y^{-\frac{1}{2}} e^y dy.$$

We can now employ the series expansion as in the previous case and deduce the numerical value of $t_1$, for various values of $d$ and corresponding values of $v_0$. The actual time taken by an electron of class (ii.) to reach the plate is thus equal to $t = t_1 + t_1 + t_2$.

### 3 (3). *Calculation of the Conductivity.*

The value of $v_0$ thus obtained gives for the two cases the critical velocity with which the electrons must leave the grid-cylinder in order to reach the plate in the time $t$ under the influence of the impressed E.M.F. $V_m$. As indicated before it is obvious that all electrons starting with velocity greater than $v_0$ will also reach the plate within the scheduled time $t$. The total number of electrons $n$ reaching the plate will therefore be the time-interval $t$ multiplied by the

number of electrons lying within the velocity range $v_0$ to $\infty$, and is therefore given by

$$tN\sqrt{\frac{m}{2\pi KT}}\int_{v_0}^{\infty}\epsilon^{-\frac{mv_0^2}{2KT}}dv_0.$$

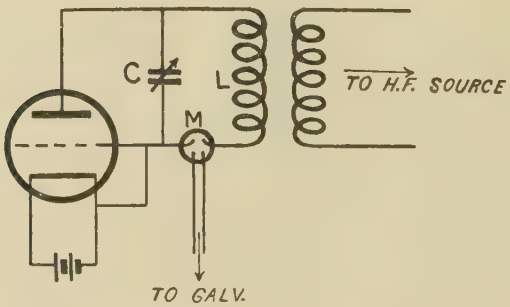
The conductivity of the grid-plate space for the high frequency impressed E.M.F. of periodicity $T=2t$ is thus proportional to $n$

$$=\frac{tN}{2t}\sqrt{\frac{m}{2\pi KT}}\int_{v_0}^{\infty}\epsilon^{-\frac{mv_0^2}{2KT}}dv_0$$

$$=\frac{N}{2}\sqrt{\frac{m}{2\pi KT}}\int_{v_0}^{\infty}\epsilon^{-\frac{mv_0^2}{2KT}}dv_0.$$

All the quantities in the expression are known except N.

Fig. 4.




Hence the fraction $\frac{n}{N}$ of the total number of electrons N reaching the plate for a particular velocity $v_0$ corresponding to a frequency $f=\frac{1}{2t}$ can be calculated. The conductivity of the grid-plate space is evidently proportional to this number.

## 4. Experiment and Result.

As mentioned previously, the resistance of the grid-plate space was found by the distuning method. The experimental arrangement is shown diagrammatically in fig. 4. The resistance of the oscillatory circuit associated with the valve, consisting of the condenser C in parallel with the

condenser formed by the grid-plate capacity, and the thermo-junction M in series with the inductance L is first determined with the filament of the valve cold. This resistance $R_1$ includes the total loss resistance in the circuit such as losses in the condenser, in the valve legs, high-frequency resistance of the coil, the resistance of the thermo-junction, etc. The resistance is again measured ($R_2$) with the filament lighted. The resistance $R_2$ consists of the previous resistance $R_1$, together with the equivalent series resistance $r$ due to the conductivity acquired by the plate-grid space owing to the presence of the electrons. The difference $R_2-R_1$, which gives the equivalent series resistance, is related to the conductivity $\frac{1}{r'}$ of the grid-plate space by the relation

$$r=\frac{1}{r'}\cdot\frac{1}{\omega^2C^2}.$$

If a condenser is shunted by an ordinary non-inductive resistance of value $r'$, then the value as determined by the above method for various frequencies is found to be sensibly constant. But, for the shunt-resistance formed by the grid-plate space of a glowing valve, this resistance is not constant, but varies in accordance with the formulæ derived in the previous section.

The distuning method of measuring resistance of an oscillatory circuit * consists, as is well known, in noting the value of the current, first, while it is in resonance with the impressed high-frequency oscillations, and again, while the reactance of the circuit has been varied by slightly distuning the circuit.

The formula employed in the calculation was

$$R=\frac{153\lambda_m(C_2-C_1)}{C_0^2},$$

where R is the total resistance of the circuit, $\lambda_m$ is the resonant wave-length of the circuit, $C_0$ the capacity for resonance condition, and $C_2$ and $C_1$ are the altered values of the capacity, respectively greater and less than $C_0$, to bring the resonant current to ·71 of its original maximum value.

Table I. shows the proportionate number of electrons reaching the plate for different frequencies, *i. e.*, the proportionate conductivity of the plate-grid space for different

* E. B. Moullin, 'Radio Frequency Measurements,' 2nd ed. p. 276 (1931). (Charles Griffin & Co.)

wave-lengths of the high-frequency oscillating field. Table II. shows the values of the resistance of the plate-grid space, obtained experimentally for different wave-lengths.

Curve A (fig. 5) shows the relation between wave-length and the resistance, obtained by calculation. Curve B shows

Table I.

| Frequency. | $\frac{n}{N}$ Conductivity ×const. | Wave-length ($\lambda$) (metres). | $\frac{1}{n}$ Resistance ×const. |
|---|---|---|---|
| $6{\cdot}8 \times 10^7$ | $1{\cdot}683 \times 10^{-8}$ | 4·41 | $\cdot 594 \times 10^8$ |
| $5{\cdot}3 \times 10^7$ | $1{\cdot}994 \times 10^{-8}$ | 5·66 | $\cdot 501 \times 10^8$ |
| $4{\cdot}21 \times 10^7$ | $2{\cdot}43 \times 10^{-8}$ | 7·11 | $\cdot 411 \times 10^8$ |
| $3{\cdot}77 \times 10^7$ | $2{\cdot}636 \times 10^{-8}$ | 7·95 | $\cdot 379 \times 10^8$ |
| $3{\cdot}2 \times 10^7$ | $3{\cdot}135 \times 10^{-8}$ | 9·37 | $\cdot 318 \times 10^8$ |
| $2{\cdot}8 \times 10^7$ | $3{\cdot}484 \times 10^{-8}$ | 10·71 | $\cdot 287 \times 10^8$ |
| $1{\cdot}026 \times 10^7$ | $2{\cdot}116 \times 10^{-8}$ | 29·2 | $\cdot 236 \times 10^8$ |
| $8{\cdot}65 \times 10^6$ | $2{\cdot}214 \times 10^{-8}$ | 34·6 | $\cdot 225 \times 10^8$ |
| $8{\cdot}3 \times 10^6$ | $2{\cdot}36 \times 10^{-8}$ | 36·1 | $\cdot 220 \times 10^8$ |

Table II.

| Frequency. | Wave-length ($\lambda$) (metres). | Resistance (megohm). |
|---|---|---|
| $6 \times 10^7$ | 5 | ·61 |
| $3{\cdot}2 \times 10^7$ | 9·37 | ·361 |
| $2{\cdot}3 \times 10^7$ | 13 | ·31 |
| $1{\cdot}75 \times 10^7$ | 17·1 | ·280 |
| $1{\cdot}2 \times 10^7$ | 25 | ·255 |
| $1 \times 10^7$ | 30 | ·251 |

the same relation obtained by actual measurement. In order to compare the two curves, the values of the resistance obtained experimentally were multiplied by a suitable constant. It will be seen that the two curves drawn to the same scale agree with each other very satisfactorily.

As a check on the measurements, the resistance of a non-inductive grid-leak of approximately the same value as that

of the plate-grid space was measured simultaneously for the two extreme frequencies. The results (Table III.) show that the high-frequency resistance increases but slightly.

It may be remarked here that calculations show that for time-intervals greater than those corresponding to 36 metres,

Fig. 5.

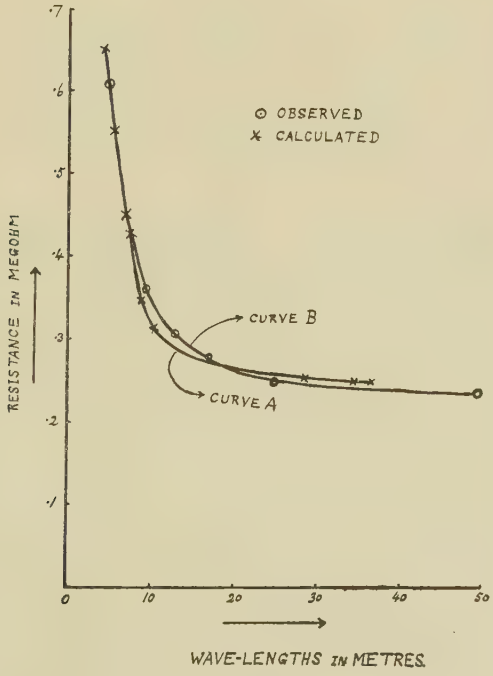



TABLE III.

| Frequency. | Wave-length (λ) (metres). | Resistance (megohm). |
|---|---|---|
| $5{\cdot}3\times10^7$ | 5·6 | ·226 |
| $8{\cdot}1\times10^6$ | 37 | ·220 |

the number of electrons reaching the plate under the action of the applied E.M.F. is equal to N. As a consequence the resistance should remain unaltered for all wave-lengths greater than 36 metres. Measurements show, however, that the resistance does increase, though to a smaller extent, for

wave-lengths greater than 36 metres. This is probably due to the fact that we have not been able to take into account the electrons of class (iii.) for which we have not found it possible to calculate the time of flight.

## 5. Conclusion.

Experiments have been made to determine the variation of conductivity of the plate-grid space of a triode valve within the frequency range $10^7$ to $6 \times 10^7$ cycles per sec. The results show that in going from the lower to the higher frequency, the conductivity decreases by more than 100 per cent. A theory of the variation of resistance has been developed on the following lines :—

It is assumed that the electrons within the plate-grid space have a Maxwellian distribution of velocity. An alternating field applied between the plate and the grid produces in the electrons an oscillatory motion, which is superimposed on their original Maxwellian velocity. The chance of an electron reaching the plate and thereby contributing towards the conductivity of the plate-grid space thus depends both on the time during which the impressed E.M.F. acts in the direction grid to plate and also on the electron's initial thermic velocity of Maxwellian type. It is possible to find a critical velocity $v_0$ with which the electron must be moving initially in order that the applied field acting for a small interval of time $t$ is just able to carry the electron to the plate. All the electrons which had been moving with a velocity greater than $v_0$ will also reach the plate within this time $t$, which is half the time-period T of the applied high frequency E.M.F. The number of electrons reaching the plate per unit time (*i. e.*, the conductivity) for a particular frequency $f$ can thus be found by calculating the critical velocity $v_0$ corresponding to the time-interval $\frac{1}{2f}$, and determining the number of electrons lying in the velocity range $v_0$ to $\infty$. Values of the conductivity calculated on the above hypothesis agree well with the values found experimentally.

Wireless Laboratory,
University College of Science,
Calcutta.
October 13, 1931.

# CIV. *The Damping of Pendulums immersed in a Viscous Fluid.*

*By* E. Tyler, *DSc., F.Inst.P., Senior Lecturer, Physics Department, College of Technology, Leicester* *.

## Introduction.

IN an earlier paper †, entitled "The Free Damping of an Aerofoil Pendulum immersed in a Fluid," it has been shown that for an object of aerofoil section attached to the lower end of a vertical reed and free to vibrate with part of it submerged in a stationary fluid, the damping experienced is mainly influenced by the area of projection of the immersed part, when in the undisturbed position, upon a vertical plane perpendicular to that in which the vibrations take place. When such a pendulum is immersed in a steady stream of fluid and free to vibrate at right angles to the flow, provided the speed is sufficient to produce the disengagement of alternate vortices in its wake, as a consequence of such eddy formation, it will be acted upon by periodic forces tending to make it vibrate across the stream. Such a phenomenon was shown in effect by Rayleigh ‡, using a cylindrical pendulum in water, and Riabouchinsky § has subsequently confirmed the condition for maintained vibrations of a pendulum in a current of air, namely, resonance when its natural period coincides with that of the production of pairs of vortices.

This phenomena provides a method for measuring the frequency of vortex formation at varying Reynolds numbers, which many experimenters ‖ have employed.

An attempt to evaluate the strength of the impulses given to a cylinder in water by the shedding of alternate vortices has recently been made by Thom ¶, who, by measuring the amplitude of maintained vibration of the cylinder, together with the corresponding fluid speed, has computed the cross-force in terms of the drag coefficient, the fluid speed, and

* Communicated by the Author.

† Tyler, Phil. Mag. xi. (Suppl.) Feb. 1931.

‡ Phil. Mag. xxix. p. 433 (1915).

§ L'Aerophile, 1st Jan. 1911; also Internat. Air Congress, London, 1923, p. 282.

‖ Kruger and Lauth, *Ann. d. Phys.* xliv. p. 801 (1914); Richardson, Proc. Phys. Soc. xxxvi. (1924); Tyler, Phil. Mag., June 1931; Dupin, *Comp. Rend.* cxci. no. 12 (Sept. 1930); Dupin and Crausse, *Comp. Rend.* cxcii. no. 12 (March 1931); Dupin and Teissie-Solier, *Comp. Rend.* cxcii. April 27, 1931; Crausse and Baubiac, *Comp. Rend.* no. 24, June 15th, 1931; also *loc. cit.* no. 22, June 1st, 1931.

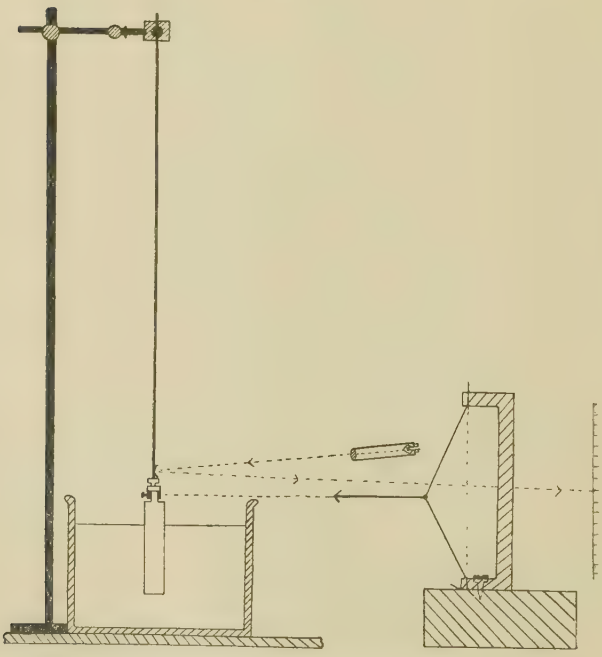
¶ Aero. Res. Comm., Rep. and Mem. no. 1373, Dec. 1930; Phil. Mag., Aug. 1931.

the maximum velocity attained by the cylinder in an alternation. In view of this resonance criterion, it is to be expected that the free vibrations of any pendulum with damping may vary considerably with and without flow, and it is with this primary object that the following investigation appertaining to cylinders and inclined plates in different liquids has been attempted.

The problem was attacked from two directions, namely :—

1. Immersion of cylinders and plates in different liquids at varying depths, and examination of the free damping of the pendulum motion without flow.
2. Repetition of 1, with cylinders in a stream of water at different speeds of flow.

Fig. 1.



Experimental Arrangement.

The arrangement of suspending the models was similar to that used for the aerofoil pendulum, in which a long steel vertical blade was clamped at its upper end and carried a small metal clamp at its lower, into which was mounted either the cylinder or plate under test (see fig. 1). Rotation

of the clamp in a horizontal plane enabled the plate attached to it to be set at any required angle with the plane of vibration, and the attachment of a mirror to the lower end of the reed permitted, with the aid of a lamp and scale, the pendulum motion to be followed clearly. Throughout the whole work the distance of the lamp and scale was kept fixed with respect to the mirror, namely, 100 cm.

In the first investigation using the cylinders (wood or ebonite), each in turn was immersed vertically to a fixed depth in the given liquid contained in a large circular tank mounted on an adjustable table, and then displaced about 5° from its position of rest.

When the amplitude of vibration had reached a suitable value for observation on the scale, successive amplitudes of swings on the same side of the zero were observed at intervals of two vibrations, until the motion had become practically damped out. The same amplitude was reached each time before observations were made at successive intervals. A similar process was effected repeating with a brass plate in water inclined at varying angles to the plane of vibration and at different depths of immersion.

In the second investigation a water-channel ($150 \times 17 \times 25$ cm.) of the return-flow type was employed, actuated at one end by a motor-driven paddle. The immersed cylinder was set vibrating with and without flow, and the corresponding decay curves obtained in the same way as before, the speed of flow being evaluated by timing the movement of dust particles over the water surface. An electrically heated nickel wire of diameter ·002″, carrying a current of 1 amp., and forming one arm of a Wheatstone bridge, served also in this respect, the wire being previously calibrated on a whirling arm.

The possibility of the walls of the channel influencing the vibrations must not be overlooked, and to test this point, results using the same cylinder both in the large tank and water-channel were compared for equal immersion depths with no flow. In both methods good consistency was obtained, leaving little doubt for any serious discrepancy due to channel-wall interference. The drag influence due to the bottom of the channel was also found negligible with the depths used.

### Measurement of Drag Coefficients.

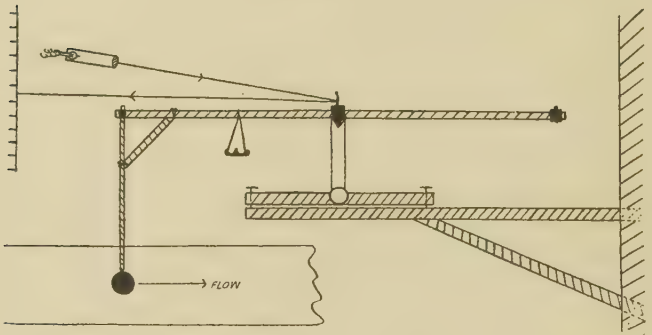
As it was desirous of knowing these coefficients for the cylinders, an ordinary laboratory balance with extended

arms was employed (see fig. 2), provision being made for mounting the cylinders horizontally and at right angles to the stream, each cylinder being in complete immersion.

Computation of the drag forces was effected by adjusting weights in the small scale pan provided on one arm of the balance. A small correction had to be applied to allow for the drag on the suspension-rod, this being known from previous calibration without the attachment of the cylinder.

In order to measure the linear displacement of the cylinders, two fine pointers were attached, one at the bottom and one half-way up the middle of each cylinder, and their displacements from the vertical determined for various scale-reading deflexions.

Fig. 2.




The differences in the observations of both pointers was so small as to justify the assumption made in the theory, namely, the cylinder as a whole moves with the average displacement of the mid-point at its middle section.

Furthermore, the desirability of measuring the restoring force per unit displacement at the lower end of the blade, or at the middle section and mid-point of the cylinder, necessitated the use of some form of balance capable of operating horizontally and at the same time sensitive to small forces. Fig. 1 shows such an arrangement used.

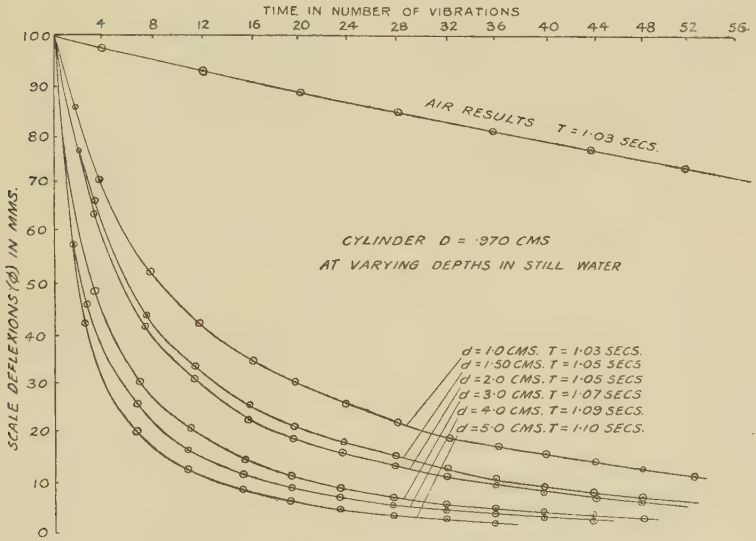
The horizontal displacement of the middle point of a stretched piece of rubber fixed vertically between two clamps, and connected either to the mid-point of the cylinder by a piece of thread or to the lower end of the blade sufficed to yield values of the restoring forces, in terms of the displacements, after calibration had been effected with the rubber-cord in the horizontal position and

weights added to a small scale pan hanging vertically from its mid-point.

## Results without Flow.

Fig. 3 exhibits a typical set of decay curves for a cylinder of diameter ·970 cm. immersed in water without flow at different depths, and fig. 4 results for a plate at constant depth, also without flow, but at varying angles of inclination to the plane of vibration.

Fig. 3.


TIME IN NUMBER OF VIBRATIONS
4 8 12 16 20 24 28 32 36 40 44 48 52 56
100 90 80 70 60 50 40 30 20 10 0
SCALE DEFLEKIONS (Ø) IN MMS.
AIR RESULTS T = 1·03 SECS.
CYLINDER D = ·970 CMS
AT VARYING DEPTHS IN STILL WATER
d = 1·0 CMS. T = 1·03 SECS.
d = 1·50 CMS. T = 1·05 SECS.
d = 2·0 CMS. T = 1·05 SECS.
d = 3·0 CMS. T = 1·07 SECS.
d = 4·0 CMS. T = 1·09 SECS.
d = 5·0 CMS. T = 1·10 SECS.

The corresponding log (amplitude) time graphs are shown in figs. 5 & 6, from which it will be observed the vibrations are of two distinct types: (1) with large amplitudes of swing the logarithmic decrement of the damping is not constant, and may be referred as turbulent damping, and (2) with smaller amplitudes there is constancy of the logarithmic decrement, corresponding to the linear parts of the graphs; this we shall regard as streamline damping.

Analogous effects have been observed by Hoare*, Brindley, and Emmerson, for spherical pendulums suspended by fine wires in a viscous medium.

* Phil. Mag., Aug. 1931.

Fig. 4.

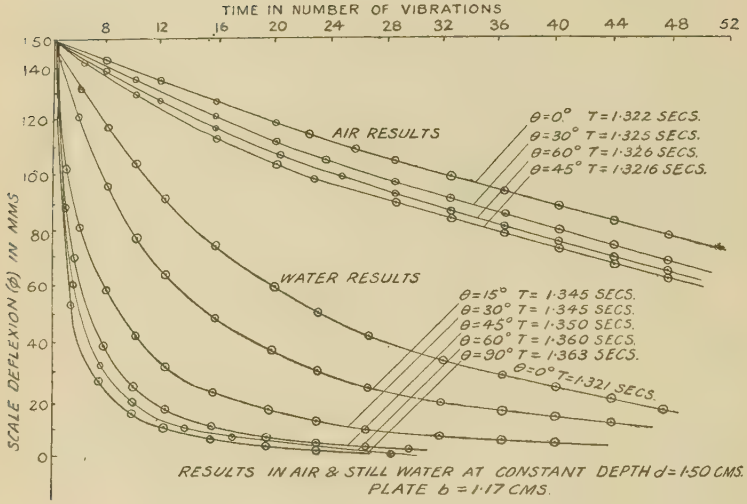

TIME IN NUMBER OF VIBRATIONS
8 12 16 20 24 28 32 36 40 44 48 52
150 140 120 100 80 60 40 20 0
SCALE DEFLEXION (Φ) IN MMS
AIR RESULTS
θ=0° T=1·322 SECS.
θ=30° T=1·325 SECS.
θ=60° T=1·326 SECS.
θ=45° T=1·3216 SECS.
WATER RESULTS
θ=15° T=1·345 SECS.
θ=30° T=1·345 SECS.
θ=45° T=1·350 SECS.
θ=60° T=1·360 SECS.
θ=90° T=1·363 SECS.
θ=0° T=1·321 SECS.
RESULTS IN AIR & STILL WATER AT CONSTANT DEPTH d=1·50 CMS.
PLATE b=1·17 CMS.

Fig. 5.

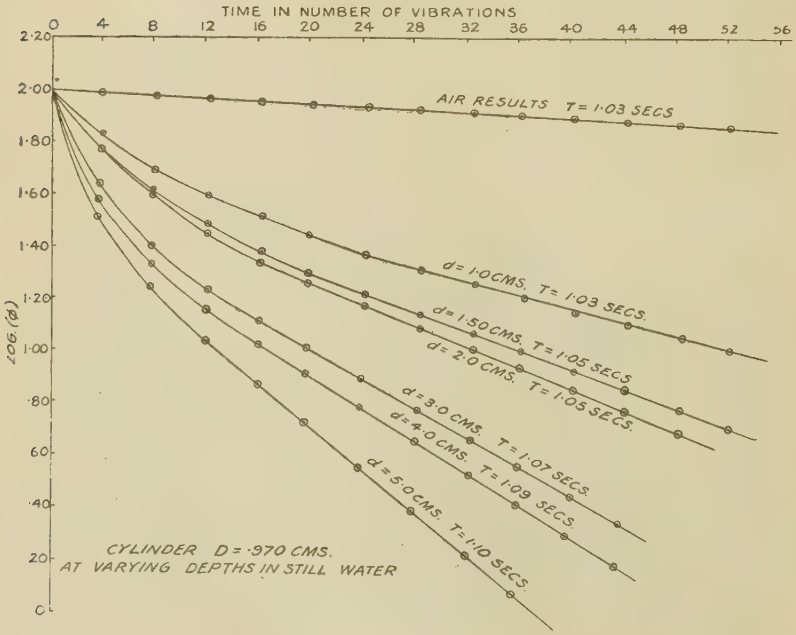

TIME IN NUMBER OF VIBRATIONS
0 4 8 12 16 20 24 28 32 36 40 44 48 52 56
2·20 2·00 1·80 1·60 1·40 1·20 1·00 ·80 ·60 ·40 20 0
LOG (Φ)
AIR RESULTS T=1·03 SECS
d=1·0 CMS. T=1·03 SECS.
d=1·50 CMS. T=1·05 SECS
d=2·0 CMS. T=1·05 SECS.
d=3·0 CMS. T=1·07 SECS.
d=4·0 CMS. T=1·09 SECS.
d=5·0 CMS. T=1·10 SECS.
CYLINDER D=·970 CMS.
AT VARYING DEPTHS IN STILL WATER

Fig. 6.

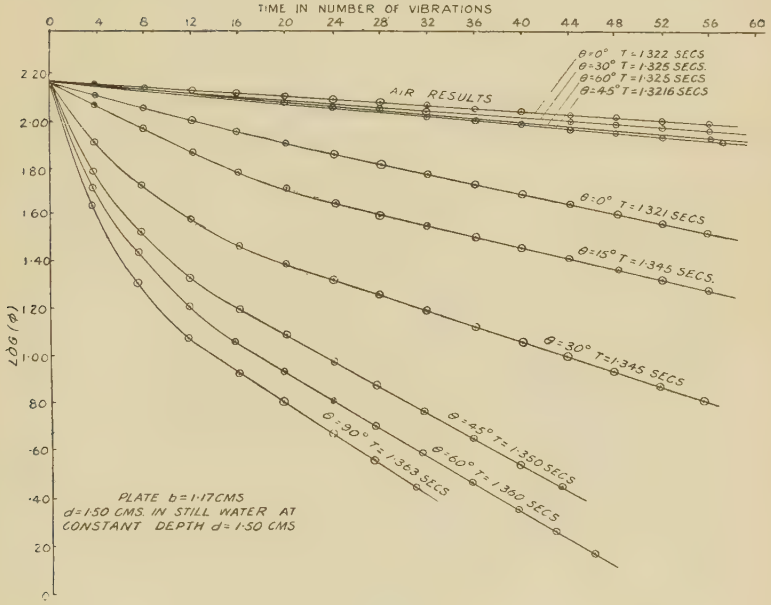
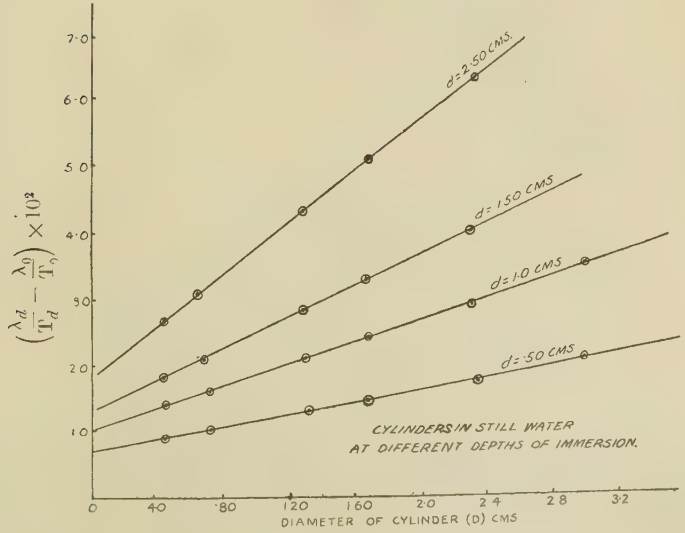

TIME IN NUMBER OF VIBRATIONS
0 4 8 12 16 20 24 28 32 36 40 44 48 52 56 60
LOG (θ)
2·20
2·00
1·80
1·60
1·40
1·20
1·00
·80
·60
·40
·20
0
AIR RESULTS
θ=0° T=1·322 SECS
θ=30° T=1·325 SECS.
θ=60° T=1·325 SECS
θ=45° T=1·3216 SECS
θ=0° T=1·321 SECS
θ=15° T=1·345 SECS.
θ=30° T=1·345 SECS
θ=45° T=1·350 SECS
θ=60° T=1·360 SECS
θ=90° T=1·363 SECS
PLATE b=1·17CMS
d=1·50 CMS. IN STILL WATER AT
CONSTANT DEPTH d=1·50 CMS

Fig. 7.


$\left(\frac{\lambda_d}{T_d} - \frac{\lambda_0}{T_0}\right) \times 10^3$
7·0
6·0
5·0
4·0
3·0
2·0
1·0
d=2·50 CMS.
d=1·50 CMS
d=1·0 CMS
d=·50 CMS
CYLINDERS IN STILL WATER
AT DIFFERENT DEPTHS OF IMMERSION.
0 ·40 ·80 1·20 1·60 2·0 2·4 2·8 3·2
DIAMETER OF CYLINDER (D) CMS

A summary of results for cylinders and plates of different diameters immersed in water are included in figs. 7, 8, 9, & 10

Fig. 8.

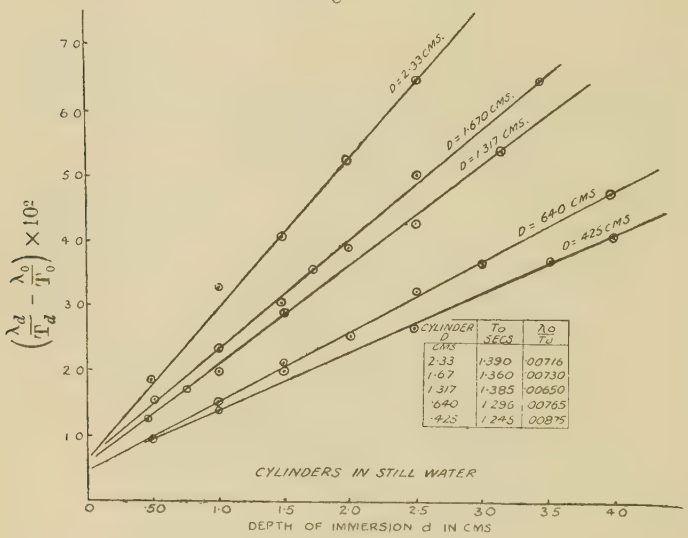
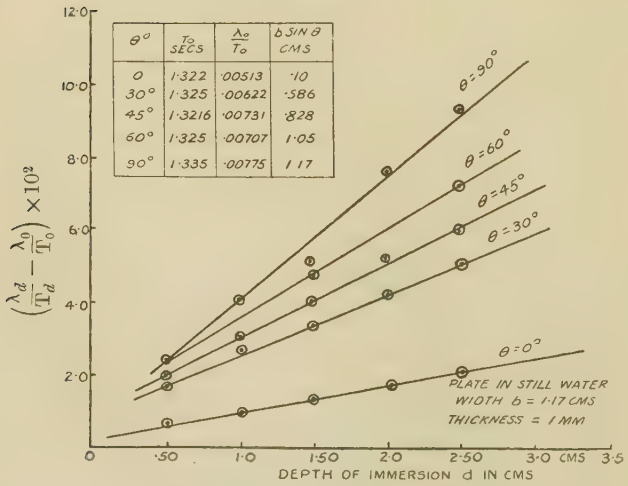

$\left(\frac{\lambda_d}{T_d} - \frac{\lambda_0}{T_0}\right) \times 10^2$
D = 2·33 CMS.
D = 1·670 CMS.
D = 1·317 CMS.
D = ·640 CMS
D = ·425 CMS
CYLINDER D CMS | To SECS | λo/To
2·33 | 1·390 | ·00716
1·67 | 1·360 | ·00730
1·317 | 1·385 | ·00650
·640 | 1·296 | ·00765
·425 | 1·245 | ·00875
CYLINDERS IN STILL WATER
DEPTH OF IMMERSION d IN CMS

Fig. 9.


θ° | To SECS | λo/To | b SIN θ CMS
0 | 1·322 | ·00513 | ·10
30° | 1·325 | ·00622 | ·586
45° | 1·3216 | ·00731 | ·828
60° | 1·325 | ·00707 | 1·05
90° | 1·335 | ·00775 | 1·17
$\left(\frac{\lambda_d}{T_d} - \frac{\lambda_0}{T_0}\right) \times 10^2$
θ = 90°
θ = 60°
θ = 45°
θ = 30°
θ = 0°
PLATE IN STILL WATER
WIDTH b = 1·17 CMS
THICKNESS = 1 MM
DEPTH OF IMMERSION d IN CMS

and Table I., the logarithmic decrements being deduced in each case from the linear parts of the graphs similar in character to those in fig. 5.

In order to test the influence on the free damping with respect to a change in viscosity and density of the fluid solutions of treacle, aniline, alcohol, and paraffin were used with

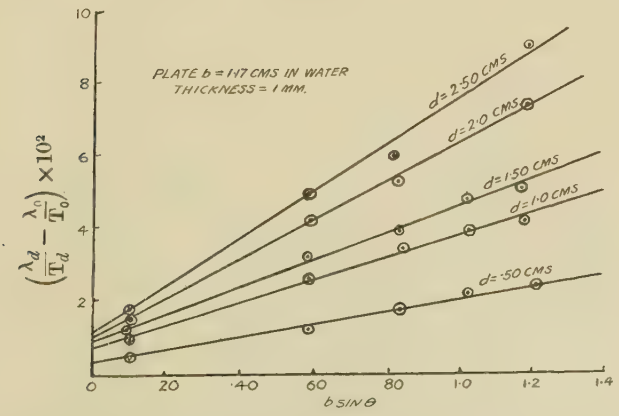
TABLE I.

Plate $b = 1{\cdot}17$ cm. Thickness 1 mm.

Depth immersed in water $= 1{\cdot}50$ cm. ($d$).

| Angle of inclination, $\theta°$. | Period, $T_d$ sec. | $\frac{\lambda_0}{T_0}$. | $\frac{\lambda_d}{T_d}$. | $\left(\frac{\lambda_d}{T_d} - \frac{\lambda_0}{T_0}\right)$. | $b \sin\theta$. |
|---|---|---|---|---|---|
| 0 | 1·321 | ·00513 | ·01635 | ·01122 | ·10 |
| 15 | 1·345 | ·00560 | ·01845 | ·01285 | ·302 |
| 30 | 1·345 | ·00622 | ·02850 | ·02228 | ·586 |
| 45 | 1·350 | ·00731 | ·04890 | ·04159 | ·825 |
| 60 | 1·360 | ·00707 | ·05180 | ·04473 | 1·05 |
| 90 | 1·357 | ·00750 | ·0570 | ·0495 | 1·10 |

Fig. 10.




a cylinder of diam. $= 1{\cdot}317$ cm. and period about 1·300 sec., and the data incorporated in figs. 11 & 12 and Table II. demonstrate the marked effect those properties have on the logarithmic decrement. Further reference is made later.

Fig. 11.

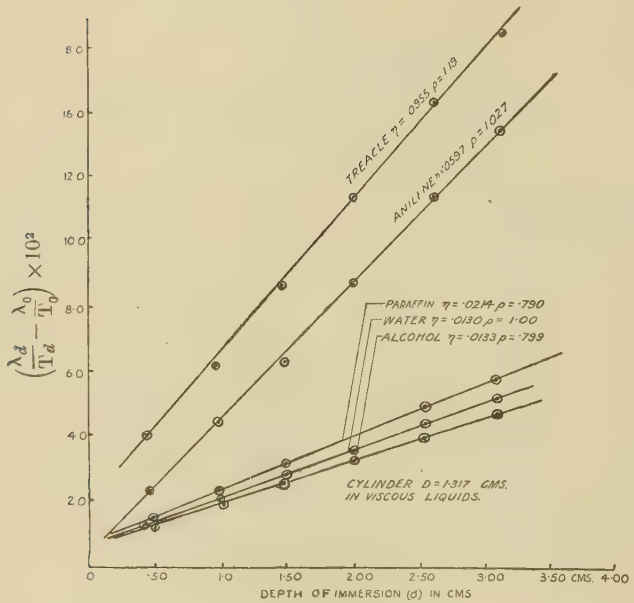

TREACLE η = 0·855, ρ = 1·19
ANILINE η = 0·037 ρ = 1·027
PARAFFIN η = 0·014 ρ = 0·790
WATER η = 0·0130, ρ = 1·00
ALCOHOL η = 0·0133, ρ = 0·799
CYLINDER D = 1·317 CMS. IN VISCOUS LIQUIDS.
$\left(\frac{\lambda_d}{T_d} - \frac{\lambda_0}{T_0}\right) \times 10^2$
DEPTH OF IMMERSION (d) IN CMS
0 ·50 1·0 1·50 2·00 2·50 3·00 3·50 CMS. 4·00
2·0 4·0 6·0 8·0 10·0 12·0 16·0 8·0

Fig. 12.

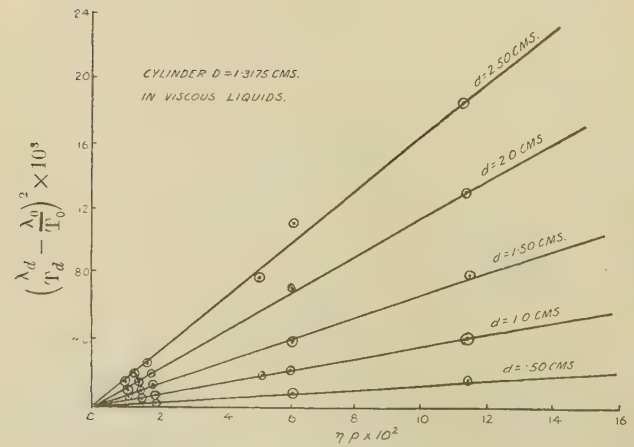

CYLINDER D = 1·3175 CMS. IN VISCOUS LIQUIDS.
d = 2·50 CMS.
d = 2·0 CMS.
d = 1·50 CMS.
d = 1·0 CMS.
d = ·50 CMS.
$\left(\frac{\lambda_d}{T_d} - \frac{\lambda_0}{T_0}\right)^2 \times 10^3$
$\eta\rho \times 10^2$
0 2 4 6 8 10 12 14 16
4 8 12 16 20 24

TABLE II.

Cylinder D=1·317 cm. in different liquids at rest.

| Liquid. | $d$. cm. | Period. $T_d$ sec. | $\frac{\lambda_0}{T_0}$. | $\frac{\lambda_d}{T_d}$. | $\left(\frac{\lambda_d}{T_d}-\frac{\lambda_0}{T_0}\right)^2 \times 10^4$. |
|---|---|---|---|---|---|
| Aniline. $\eta$= ·0517 gm./cm./sec. $\rho$=1·027 gm./c.c. | 0 | 1·385 | ·0063 | ·0063 | 0 |
| | ·50 | 1·385 | | ·0310 | 6·10 |
| | 1·00 | 1·400 | | ·0520 | 20·90 |
| | 1·50 | 1·417 | | ·0635 | 33·40 |
| | 2·00 | 1·442 | | ·0988 | 85·50 |
| | 2·50 | 1·460 | | ·1120 | 111·5 |
| Alcohol. $\eta$=·0135 gm./cm./sec. $\rho$=·799 gm./c.c. | 0 | 1·385 | ·0063 | ·0063 | 0 |
| | ·50 | 1·383 | | ·0203 | 1·96 |
| | 1·00 | 1·400 | | ·0259 | 3·86 |
| | 1·50 | 1·410 | | ·0340 | 7·70 |
| | 2·00 | 1·422 | | ·0399 | 11·28 |
| | 2·50 | 1·425 | | ·0451 | 15·00 |
| Paraffin. $\eta$=·0214 gm./cm./sec. $\rho$=·790 gm./c.c. | 0 | 1·380 | ·0061 | ·0061 | 0 |
| | ·50 | 1·400 | | ·0210 | 2·22 |
| | 1·0 | 1·410 | | ·0320 | 6·70 |
| | 1·50 | 1·425 | | ·0376 | 9·90 |
| | 2·00 | 1·430 | | ·0429 | 13·52 |
| | 2·50 | 1·430 | | ·0524 | 21·50 |
| Treacle. $\eta$= ·0955 gm./cm./sec. $\rho$=1·19 gm./c.c. | 0 | 1·375 | ·0061 | ·0061 | 0 |
| | ·50 | 1·395 | | ·0505 | 19·75 |
| | 1·00 | 1·402 | | ·0681 | 38·50 |
| | 1·50 | 1·450 | | ·0915 | 73·0 |
| | 2·00 | 1·485 | | ·1208 | 131·5 |
| | 2·50 | 1·486 | | ·1421 | 185·0 |
| Water. $\eta$= ·013 gm./cm./sec. $\rho$=1·0 gm./c.c. | 0 | 1·385 | ·0065 | ·0065 | 0 |
| | ·50 | 1·397 | | ·0225 | 2·55 |
| | 1·00 | 1·405 | | ·0274 | 4·36 |
| | 1·50 | 1·410 | | ·0366 | 9·05 |
| | 2·00 | 1·430 | | .0450 | 14·85 |
| | 2·50 | 1·450 | | ·0485 | 17·62 |
| | 3·00 | 1·460 | | ·0505 | 19·40 |

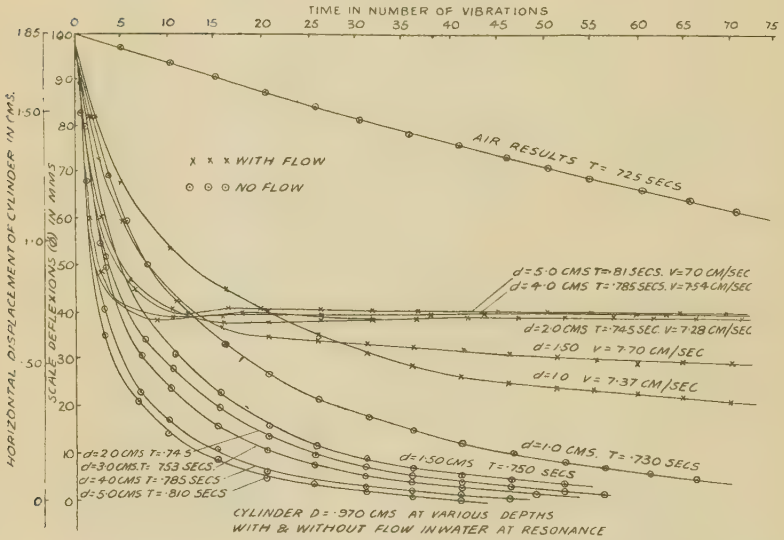
## Results with Flow.

Two effects were examined:

(1) Maintaining steady flow at resonant condition, free damping curves for cylinders were obtained at different depths of immersion and also different periods in water.

(2) At a constant depth for a cylinder, free damping curves at varying speeds of flow.

Fig. 13.


TIME IN NUMBER OF VIBRATIONS
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75
HORIZONTAL DISPLACEMENT OF CYLINDER IN CMS.
SCALE DEFLEXIONS (θ) IN MMS
185 1·50 1·0 ·50 0
100 90 80 70 60 50 40 30 20 10 0
x x x WITH FLOW
o o o NO FLOW
AIR RESULTS T= 725 SECS
d=5·0 CMS T=·81 SECS. V=7·0 CM/SEC
d=4·0 CMS T=·785 SECS. V=7·54 CM/SEC
d=2·0 CMS T=·745 SEC V=7·28 CM/SEC
d=1·50 V=7·70 CM/SEC
d=1·0 V=7·37 CM/SEC
d=1·0 CMS. T=·730 SECS
d=1·50 CMS T=·750 SECS
d=2·0 CMS T=·74 5
d=3·0 CMS. T=·753 SECS
d=4·0 CMS T=·785 SECS
d=5·0 CMS T=·810 SECS
CYLINDER D=·970 CMS AT VARIOUS DEPTHS
WITH & WITHOUT FLOW IN WATER AT RESONANCE

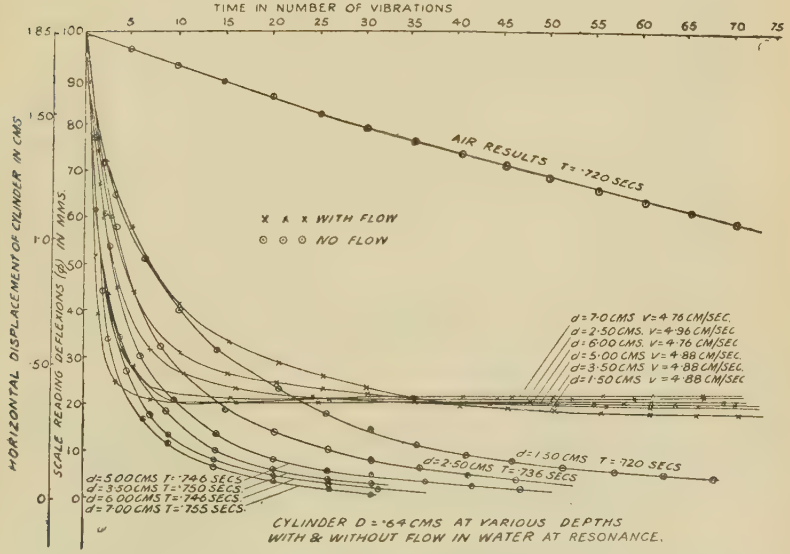
With regard to the influence of flow on the free damping under these conditions, this is clearly exhibited in figs. 13, 14, 15, & 16, which contain results for cylinders D=·970, ·64, and 1·317 cm. respectively. Figs. 17 & 18 show the effect of variation of the speed of flow on the damping at constant depth of immersion.

It is of interest to notice the quicker response to maintained vibrations at resonance speed for increased depth of immersion, which is to be expected from hydrodynamical conditions of the forces in operation on the cylinder by the detachment of a single vortex, despite the greater damping and increase of resistance produced by the fluid.

Typical maintained vibration responses at various speeds of flow for cylinders of diameter D ·970 cm. and 1·317 cm. are shown in figs. 19 & 20, similar in character to the results obtained by Thom, and it will be observed the resonance response is more critical with the cylinder of shorter period (fig. 20).

The phenomenon of beats could be easily recognized, and this effect indirectly served as a means of indicating when resonance speeds had been reached, namely, non-existence

Fig. 14.


TIME IN NUMBER OF VIBRATIONS
HORIZONTAL DISPLACEMENT OF CYLINDER IN CMS
SCALE READING DEFLEXIONS (θ) IN MMS
AIR RESULTS T=·720 SECS
X X X WITH FLOW
⊙ ⊙ ⊙ NO FLOW
d=7·0 CMS V=4·76 CM/SEC.
d=2·50 CMS. V=4·96 CM/SEC.
d=6·00 CMS. V=4·76 CM/SEC.
d=5·00 CMS V=4·88 CM/SEC.
d=3·50 CMS V=4·88 CM/SEC.
d=1·50 CMS V=4·88 CM/SEC
d=1·50 CMS T=·720 SECS
d=2·50 CMS T=·736 SECS
d=5·00 CMS T=·746 SECS
d=3·50 CMS T=·750 SECS
d=6·00 CMS T=·746 SECS
d=7·00 CMS T=·755 SECS
CYLINDER D=·64 CMS AT VARIOUS DEPTHS
WITH & WITHOUT FLOW IN WATER AT RESONANCE.

of beats, but an increasing amplitude of swing to a steady maintained value.

## Theory and Discussion of Results.

*Case 1.—Free streamline damping of cylinders and plates immersed in a stationary fluid at varying depths.*

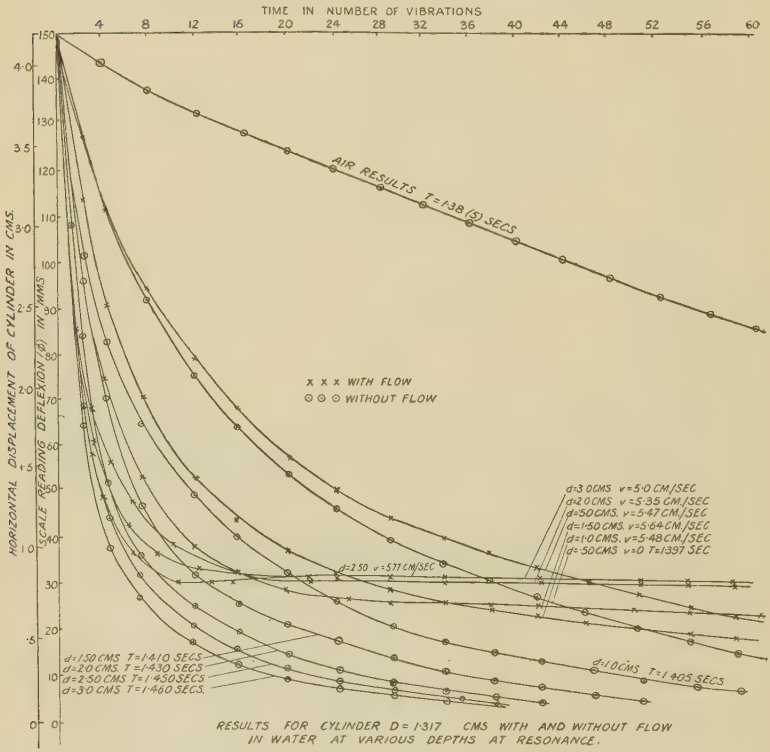
The equation of motion of a short cylinder or plate attached to the lower end of a long vertical vibrating reed in air, and executing damped harmonic motion, may be be written as

$$M\frac{d^2x}{dt^2}+k_0\frac{dx}{dt}+\mu x=0, \quad . \quad . \quad . \quad . \quad (1)$$

where M = the effective mass of the vibrating system supposed concentrated at the lower end of the blade;

$x$ = the mean linear displacement of the mid-point of the cylinder from the vertical position, this being assumed the same for the whole

Fig. 15.


TIME IN NUMBER OF VIBRATIONS
HORIZONTAL DISPLACEMENT OF CYLINDER IN CMS.
SCALE READING DEFLEXION (d) IN MMS.
AIR RESULTS T=1·38 (5) SECS
x x x WITH FLOW
⊙ ⊙ ⊙ WITHOUT FLOW
d=3·0 CMS v=5·0 CM/SEC
d=2·0 CMS v=5·35 CM/SEC
d=1·50 CMS v=5·47 CM/SEC
d=1·0 CMS v=5·64 CM/SEC
d=·50 CMS v=5·48 CM/SEC
d=0 T=1·397 SEC
d=2·50 v=5·71 CM/SEC
d=1·50 CMS T=1·410 SECS
d=2·0 CMS T=1·430 SECS
d=2·50 CMS T=1·450 SECS
d=3·0 CMS T=1·460 SECS.
d=1·0 CMS T=1·405 SECS
RESULTS FOR CYLINDER D=1·317 CMS WITH AND WITHOUT FLOW IN WATER AT VARIOUS DEPTHS AT RESONANCE.

cylinder, since the length of the blade is much longer than the cylinder and the displacement is small;

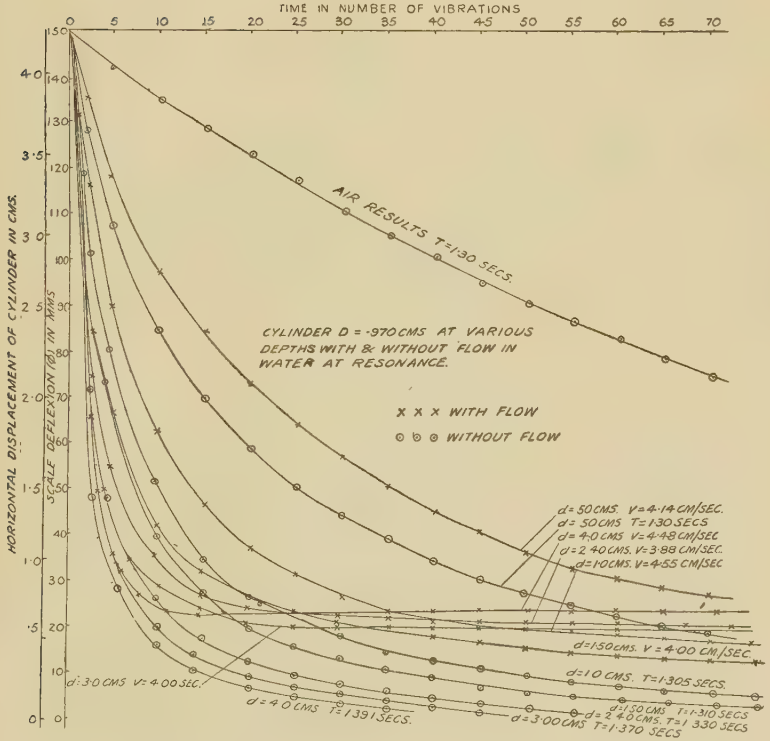
$k_0 \frac{dx}{dt}$ = the frictional damping force experienced by the cylinder or blade;

$ux$ = the restoring force exerted by the blade and cylinder dependent upon the effective mass of the system and the elasticity of the blade.

When a cylinder of diameter D cm. is immersed in a stationary fluid to a depth $d$ cm., allowing for the frictional damping of the fluid and neglecting the buoyancy effect, since it is small, the equation of motion now becomes

$$M\frac{d^2x}{dt^2}+(k_d+k_r)\cdot\frac{dx}{dt}+\mu x=0, \quad . \quad . \quad . \quad (2)$$

Fig. 16.


TIME IN NUMBER OF VIBRATIONS
HORIZONTAL DISPLACEMENT OF CYLINDER IN CMS.
SCALE DEFLEXION (D) IN MMS
AIR RESULTS T=1·130 SECS.
CYLINDER D = ·970 CMS AT VARIOUS DEPTHS WITH & WITHOUT FLOW IN WATER AT RESONANCE.
x x x WITH FLOW
o o o WITHOUT FLOW
d=·50 CMS. V=4·14 CM/SEC.
d=·50 CMS. T=1·130 SECS
d=4·0 CMS V=4·48 CM/SEC
d=2·40 CMS. V=3·88 CM/SEC.
d=1·0 CMS. V=4·55 CM/SEC
d=1·50 CMS. V=4·00 CM/SEC.
d=1·0 CMS. T=1·305 SECS.
d=·50 CMS T=1·310 SECS
d=2·40 CMS. T=1·330 SECS
d=3·00 CMS T=1·370 SECS
d=3·0 CMS V=4·00 SEC.
d=4·0 CMS T=1·391 SECS.

where $k_d\dfrac{dx}{dt}$ = the frictional damping force exerted by the fluid on the cylinder, and

$k_r\dfrac{dx}{dt}$ = the frictional damping force exerted on the blade and the remainder of the cylinder not immersed.

Fig. 17.

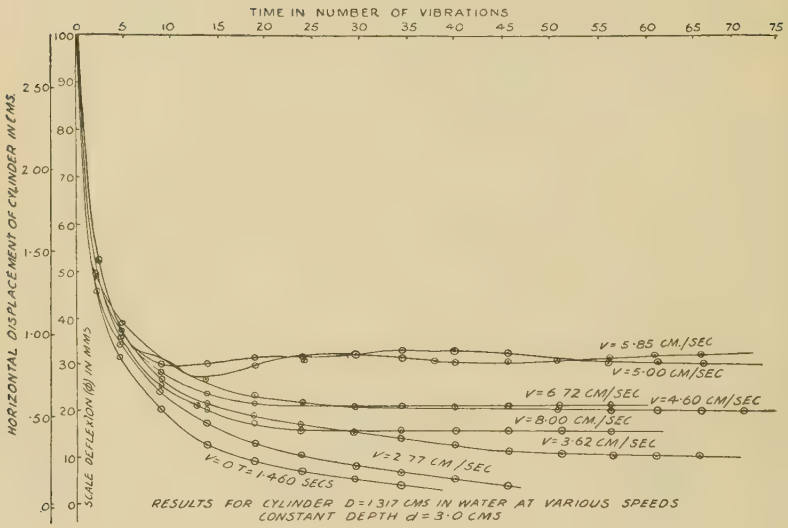
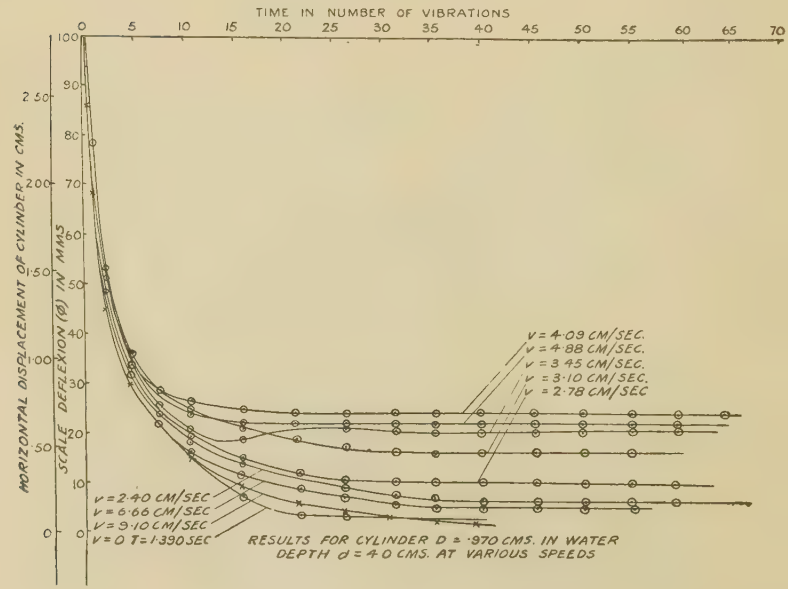

TIME IN NUMBER OF VIBRATIONS
HORIZONTAL DISPLACEMENT OF CYLINDER IN CMS.
SCALE DEFLEXION (Θ) IN MMS
V = 5·85 CM./SEC
V = 5·00 CM/SEC
V = 6·72 CM/SEC
V = 4·60 CM/SEC
V = 8·00 CM./SEC
V = 3·62 CM/SEC
V = 2·77 CM/SEC
V = 0 T = 1·460 SECS
RESULTS FOR CYLINDER D = 1·317 CMS IN WATER AT VARIOUS SPEEDS
CONSTANT DEPTH d = 3·0 CMS

Fig 18.


TIME IN NUMBER OF VIBRATIONS
HORIZONTAL DISPLACEMENT OF CYLINDER IN CMS.
SCALE DEFLEXION (Θ) IN MMS
V = 4·09 CM/SEC.
V = 4·88 CM/SEC.
V = 3·45 CM/SEC.
V = 3·10 CM/SEC.
V = 2·78 CM/SEC
V = 2·40 CM/SEC
V = 6·66 CM/SEC
V = 9·10 CM/SEC
V = 0 T = 1·390 SEC
RESULTS FOR CYLINDER D = ·970 CMS. IN WATER
DEPTH d = 4·0 CMS. AT VARIOUS SPEEDS

The solution of equation (2) is

$$x = x_0 e^{-\frac{k_d+k_r}{2M}.t} . \cos\left[\sqrt{\frac{\mu}{M} - \left(\frac{k_d+k_r}{2M}\right)^2}.t + \phi\right],$$

Fig. 19.

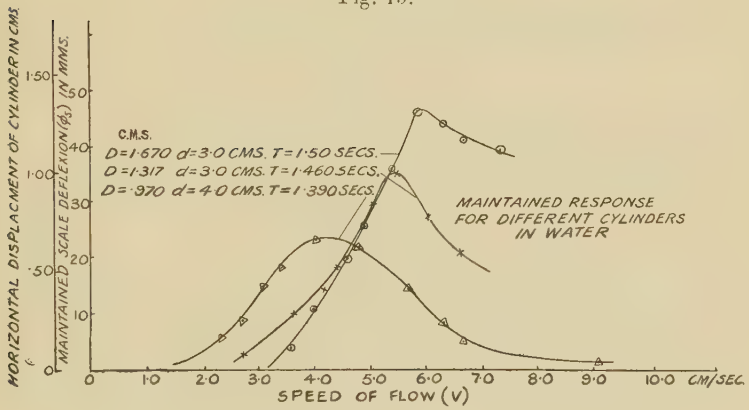
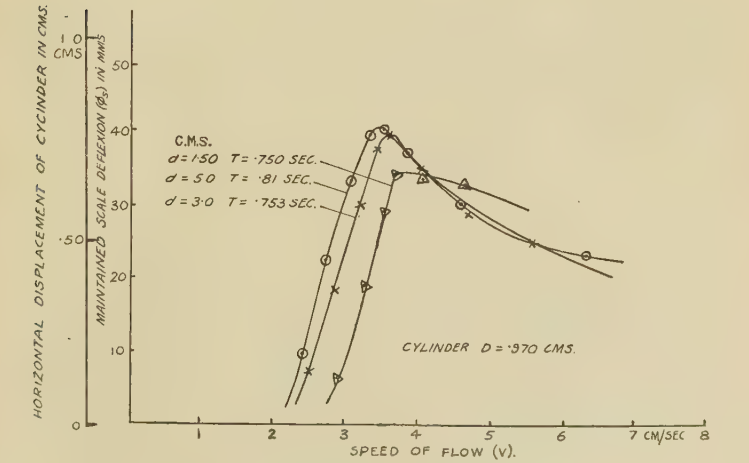

HORIZONTAL DISPLACEMENT OF CYLINDER IN CMS.
MAINTAINED SCALE DEFLEXION ($\theta_3$) IN MMS.
C.M.S.
D=1·670 d=3·0 CMS. T=1·50 SECS.
D=1·317 d=3·0 CMS. T=1·460 SECS.
D=·970 d=4·0 CMS. T=1·390 SECS.
MAINTAINED RESPONSE FOR DIFFERENT CYLINDERS IN WATER
SPEED OF FLOW (V)
CM/SEC.

Fig. 20.




$x_0$ and $\phi$ being constants dependent upon the initial conditions. The maximum value of $x$ after $m$ complete vibrations from the start is given as

$$x_m = x_0 e^{-\frac{k_d+k_r}{2M}.mT_d}, \quad \ldots \quad (3)$$

where $T_d$=the period of the system for depth of immersion $d$ cm., and the successive amplitudes of swing are $x_0$, $x_1$, $x_2$, $x_3$, etc. Also

$$\log_e \frac{x_0}{x_m} = 2m.\lambda_d,$$

which, connecting with equation (3), gives

$$\lambda_d = \frac{1}{4}\frac{(k_d+k_r)}{M}.T_d. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Putting $\lambda_0$=the log dec. of the system for zero depth of immersion, and $T_0$ as the period, $k_d=0$ and $k_r$ becomes $k_0$.

$$\therefore \qquad \lambda_0 = \frac{1}{4}.\frac{k_0 T_0}{M}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

whence from equations (4) and (5),

$$\frac{\lambda_d}{T_d} - \frac{\lambda_0}{T_0} = \frac{1}{4M}(k_d+k_r-k_0) \doteq \frac{1}{4}\frac{k_d}{M},$$

since $(k_r-k_0)$ is generally small compared with $k_d$.

Assuming, now, for a given fluid that $k_d$ varies linearly with respect to the effective area of the immersed part of the cylinder projected on to a vertical plane perpendicular to the plane of vibration, suggestive from the previous results of the aerofoil pendulum, hence for a cylinder of diameter D cm. immersed to a depth $d$ cm., we may put

$$k_d = A + B.(D.d),$$

giving

$$\left(\frac{\lambda_d}{T_d} - \frac{\lambda_0}{T_0}\right)_{\text{cylinder}} = A_1 + B_1.(D.d),$$

A, $A_1$, B, $B_1$ being constants for the fluid, depending on the nature of the fluid in respect of density and viscosity.

In the case of a plate of width $b$ cm., inclined at $\theta°$ to the plane of vibration and immersed $d$ cm., we must replace $D.d$ by $bd\sin\theta$, whence

$$\left(\frac{\lambda_d}{T_d} - \frac{\lambda_0}{T_0}\right)_{\text{plate}} = A_2 + B_2(bd\sin\theta).$$

In order to account for the effect of the viscosity $\eta$ and the density $\rho$ of the fluid, since there is a linear relationship between $\left(\frac{\lambda_d}{T_d} - \frac{\lambda_0}{T_0}\right)^2$ and $\eta\rho$ (see fig. 12), we may include

these quantities as follows to express the experimental results, viz.,

$$\left(\frac{\lambda_d}{T_d}-\frac{\lambda_0}{T_0}\right)_{\text{cylinder}}=A_3+B_3(D.d)\sqrt{\eta\rho} \quad . \quad . \quad . \quad (6)$$

and

$$\left(\frac{\lambda_d}{T_d}-\frac{\lambda_0}{T_0}\right)_{\text{plate}}=A_4+B_4 b.\sin\theta\sqrt{\eta\rho}\,; \quad . \quad . \quad . \quad (7)$$

alternately we obtain

$$\left(\frac{\lambda_{d_1}}{T_{d_1}}-\frac{\lambda_{d_2}}{T_{d_2}}\right)_{\text{cylinder}}=B_3.D(d_1-d_2)\sqrt{\eta\rho} \quad . \quad . \quad . \quad (8)$$

and

$$\left(\frac{\lambda_{d_1}}{T_{d_1}}-\frac{\lambda_{d_2}}{T_{d_2}}\right)_{\text{plate}}=B_4.b.\sin\theta(d_1-d_2)\sqrt{\eta\rho}, \quad . \quad (9)$$

where $\lambda_{d_1}$ and $\lambda_{d_2}$ = the log decs. for depths of immersion $d_1$ and $d_2$ cm. and periods $T_{d_1}$ and $T_{d_2}$ respectively.

Equations (6), (7), (8), and (9) are therefore representative of the behaviour of cylinders and plates, and their validity is borne out by the data contained in figs. 11 and 12, which exhibit linear relationships between the varying quantities.

*Case II.—Free streamline damping of cylindrical pendulums in a stream of fluid under the influence of vortices.*

It is known * that the detachment of alternate vortices behind a cylinder in a fluid stream begins in the neighbourhood of $\frac{VD}{\nu}=48$. The existence of such eddies brings into operation periodic cross-forces as a result of alternating circulation about the cylinder, the sign of the force being the same as that of the vortex in formation.

Generally the net circulation round the cylinder is less than that in a single detached vortex. A knowledge of the circulation round the cylinder enables the magnitude of the cross-force to be calculated under the experimental conditions of maintained vibrations, or, *vice versa*, knowing the factors controlling maintained vibration, the circulation round the cylinder may be deduced.

Assuming K = the circulation in a single detached vortex, then $\gamma K$ is the circulation round the cylinder where $\gamma<1$. When the cylinder of diameter D cm. is immersed to a depth $d$ cm. in a fluid of density $\rho$ and speed V cm./sec., the cross-force will be $\gamma K\rho d V$.

* Tyler, Journ. Scientific Inst. vii. no. 11 (Nov. 1930).

As a tentative suggestion, assuming the cross-force is periodic and simple harmonic to a first approximation, we write

$$F = F_0 \cos pt,$$

where $F = \gamma K \rho d V$, and the vortex frequency $N = \frac{p}{2\pi}$, then the equation of motion of the system becomes

$$M\frac{d^2x}{dt^2} + (k_d + k_r)\frac{dx}{dt} + \mu x = F_0 \cos pt,$$

a solution of the particular integral being

$$x = \frac{F_0 \sin \epsilon}{p(k_d + k_r)} \cos (pt - \epsilon),$$

where $$\tan \epsilon = \frac{p}{M} \cdot \frac{k_d + k_r}{\frac{\mu}{M} - p^2},$$

and the general solution is

$$x = \frac{F_0 \sin \epsilon}{p(k_d + k_r)} \cos (pt - \epsilon) + x_0 e^{-\frac{k_d + k_r}{2M} \cdot t} \cos \left[ \sqrt{\frac{\mu}{M} - \frac{1}{4}\left(\frac{k_d + k_r}{M}\right)^2} \cdot t + \phi \right].$$

In the steady state, when the amplitude of the free vibration is small with respect to that of the forced vibration, we get

$$x_s = \frac{F_0 \sin \epsilon}{p(k_d + k_r)} \cos (pt - \epsilon)$$

or $$x_s = \frac{F_0}{M\left\{\left(\frac{\mu}{M} - p^2\right) - p^2\left(\frac{k_d + k_r}{M}\right)^2\right\}^{\frac{1}{2}}} \cos (pt - \epsilon).$$

At resonance $\frac{\mu}{M} = p^2$, giving maximum displacement

$$[x]_m = \frac{F_0}{p(k_d + k_r)} \cdot \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now since $\lambda_d = \frac{(k_d + k_r)}{4M} T_d$ and $p = 2\pi N$,

we derive the relation

$$[x]_m \frac{\lambda_d}{T_d} = \frac{F_0}{8\pi MN} = \frac{\gamma K \rho d V}{8\pi MN} \cdot \quad . \quad . \quad . \quad (11)$$

It is impossible to determine M directly, but its value may be expressed in terms of the free period of the system without damping $T_0$, together with $\mu$, the restoring force per unit displacement; thus

$$M = \frac{\mu T_0^2}{4\pi^2},$$

because $k_0$ is small compared with $\mu$.

Substituting in equation (11) gives

$$[x]_m \frac{\lambda_d}{T_d} \mu T_0^2 = \frac{\pi}{2} \cdot \frac{\gamma K \rho d V}{N}.$$

Finally, in order to account for the effect of viscosity of the fluid at low Reynolds numbers $\frac{VD}{\nu}$, it is sufficient to substitute for $\frac{V}{N}$ from the expression

$$\frac{ND}{V} = A'\phi\left(\frac{\nu}{VD}\right) = a\left(1 + b\frac{\nu}{VD}\right),$$

in which $A'$, $a$, $b$ are constants and $\nu = \frac{\eta}{\rho}$.

Experimentally $\frac{ND}{V} = \cdot 198\left(1 - 19{\cdot}7\frac{\nu}{VD}\right)$, neglecting the higher powers of $\frac{\nu}{VD}$.

Hence $$[x]_m \frac{\lambda_d}{T_d} \mu T_0^2 = 2{\cdot}52\,\pi\gamma K \rho d D\left(1 + 19{\cdot}7\frac{\nu}{VD}\right), \quad (12)$$

from which $\gamma K$ may be evaluated. Such values are included in Tables III., IV., and V.

This expression also enables one to compute comparatively the circulation around a given cylinder under maintained vibration at different periods for constant depth, since it follows that

$$\frac{[x]_{m_1} \frac{\lambda_{d_1}}{T_{d_1}} \mu_1 [T_0]_1^2}{[x]_{m_2} \frac{\lambda_{d_2}}{T_{d_2}} \mu_2 [T_0]_2^2} = \frac{K_1\left(1 + 19{\cdot}7\frac{\nu}{V_1 D}\right)}{K_2\left(1 + 19{\cdot}7\frac{\nu}{V_2 D}\right)} \doteq \frac{V_1\left(1 + 19{\cdot}7\frac{\nu}{V_1 D}\right)}{V_2\left(1 + 19{\cdot}7\frac{\nu}{V_2 D}\right)},$$

TABLE III.

Cylinder D = ·64 cm.

| $d$. cm. | $T_d$. sec. | $\frac{\lambda_d}{T_d}$. No flow. | $[x]_m$. cm. | $\mu$. gm. | V cm./sec. | $[x]_m \cdot \frac{\lambda_d}{T_d} \mu T_0^2$. | $V\rho d D^2\left(1+19{\cdot}7\,\frac{\nu}{VD}\right)$. | $\gamma K$. cm.$^2$/sec. |
|---|---|---|---|---|---|---|---|---|
| 0 | ·720 | ·0120 | | | | | | |
| 1·50 | ·720 | ·0523 | ·310 | 3·06 | 4·88 | ·0264 | 3·22 | 3·16 |
| 2·50 | ·736 | ·0750 | ·319 | 2·92 | 4·96 | 0440 | 5·46 | 3·22 |
| 3·50 | ·750 | ·0870 | ·360 | 3·71 | 4·88 | ·0505 | 7·51 | 2·60 |
| 5·00 | ·746 | ·1150 | ·375 | 3·03 | 4·88 | ·0675 | 10·73 | 2·45 |
| 7·00 | ·755 | ·1460 | ·390 | 2·92 | 4·76 | ·0866 | 14·65 | 2·25 |
| 0 | 1·00 | ·0063 | | | | | | |
| 1·0 | 1·005 | ·0251 | ·320 | 1·63 | 3·57 | ·0130 | 1·60 | 2·30 |
| 2·0 | 1·020 | ·0377 | ·340 | 1·65 | 3·78 | 0210 | 3·39 | 1·86 |
| 3·0 | 1·020 | ·0469 | ·330 | 1·67 | 3·87 | ·0258 | 5·19 | 1·53 |
| 4·0 | 1·035 | ·0645 | ·340 | 1·65 | 3·45 | ·0361 | 6·25 | 1·60 |
| 5·0 | 1·035 | ·0773 | ·340 | 1·65 | 3·92 | ·0434 | 8·75 | 1·57 |
| 6·0 | 1·051 | ·0843 | ·340 | 1·65 | 3·61 | ·0472 | 9·55 | 1·39 |
| 0 | 1·280 | ·0083 | — | — | — | — | — | — |
| 1·50 | 1·300 | ·0310 | ·250 | 1·28 | 2·73 | ·0162 | 1·89 | 1·86 |
| 2·50 | 1·300 | ·0372 | ·250 | 1·28 | 2·85 | ·0195 | 3·28 | 1·35 |
| 4·00 | 1·330 | ·0510 | ·250 | 1·28 | 2·67 | ·0267 | 4·95 | 1·15 |
| 5·50 | 1·350 | ·0690 | ·250 | 1·28 | 2·85 | ·0361 | 7·19 | 1·14 |
| 7·00 | 1·360 | ·0739 | ·250 | 1·28 | 2·74 | ·0387 | 8·85 | ·95 |

## Table IV.

Cylinder D = ·970 cm.

| $d$. cm. | $T_d$. sec. | $\frac{\lambda_d}{T_d}$. No flow. | $[x]_m$. cm. | $\mu$. gm. | V. cm./sec. | $[x]_m \cdot \frac{\lambda_d}{T_d} \mu T_0^2$. | $V\rho d D^2 \left(1+19{\cdot}7\frac{\nu}{VD}\right)$. | $\gamma K$. cm.$^2$/sec. |
|---|---|---|---|---|---|---|---|---|
| 0 | ·725 | ·0093 | | | | | | |
| 1·0 | ·730 | ·0492 | ·350 | 3·14 | 7·37 | ·0284 | 7·14 | 2·75 |
| 1·5 | ·750 | ·0665 | ·540 | 3·07 | 7·70 | ·0580 | 11·17 | 4·80 |
| 2·0 | ·745 | ·0760 | ·650 | 3·10 | 7·28 | ·0805 | 14·07 | 5·00 |
| 3·0 | ·753 | ·0885 | ·715 | 3·10 | 7·70 | ·1030 | 22·35 | 4·27 |
| 4·0 | 785 | ·1140 | ·715 | 3·10 | 7·54 | ·1330 | 29·25 | 4·13 |
| 5·0 | ·810 | ·1332 | ·715 | 3·19 | 7·00 | ·1600 | 33·90 | 4·00 |
| 0 | 1·03 | ·0057 | | | | | | |
| 1·0 | 1·03 | ·0268 | ·430 | 1·86 | 5·00 | ·0228 | 4·92 | 2·19 |
| 1·50 | 1·05 | ·0397 | ·630 | 1·90 | 5·26 | ·0503 | 7·72 | (4·14) |
| 2·00 | 1·05 | ·0423 | ·680 | 1·88 | 5·10 | ·0573 | 10·02 | 3·52 |
| 3·00 | 1·07 | ·0602 | ·685 | 1·90 | 5·04 | ·0830 | 14·82 | 3·39 |
| 4.00 | 1·09 | ·0656 | ·680 | 1·88 | 5·00 | ·0890 | 19·62 | 2·73 |
| 5.00 | 1·10 | ·0857 | ·680 | 1·88 | 5·29 | ·1160 | 16·00 | 2·84 |
| 0 | 1·300 | ·0075 | — | — | — | — | — | — |
| ·50 | 1·300 | ·0170 | — | — | — | — | — | — |
| 1·00 | 1·305 | ·0251 | ·450 | 1·33 | 4·55 | ·0254 | 4·50 | 2·41 |
| 2·50 | 1·330 | ·0354 | ·590 | 1·32 | 3·88 | ·0465 | 9·27 | 2·25 |
| 3·00 | 1·370 | ·0450 | ·610 | 1·34 | 4·00 | ·0562 | 11·85 | 2·26 |
| 4·00 | 1·391 | ·0538 | ·650 | 1·29 | 4·30 | ·0761 | 16·98 | 2·30 |

TABLE V.

Cylinder D = 1·317 cm.

| $d$. cm. | $T_d$. sec. | $\frac{\lambda_d}{T_d}$. No flow. | $[x]_m$. cm. | $\mu$. gm. | V. cm./sec. | $[x]_m \frac{\lambda_d}{T_d} \cdot \mu T_0^2$. | $V\rho d D^2 \left(1+19{\cdot}7\frac{\nu}{VD}\right)$. | $\gamma K$. cm.²/sec. |
|---|---|---|---|---|---|---|---|---|
| 0 | 1·385 | ·0061 | | | | | | |
| 1·0 | 1·405 | ·0274 | ·460 | 1·56 | 5·48 | ·0378 | 9·80 | 3·67 |
| 1·50 | 1·410 | ·0366 | ·610 | 1·54 | 5·64 | ·0660 | 15·30 | 4·44 |
| 2·00 | 1·430 | ·0450 | ·800 | 1·55 | 5·35 | ·1070 | 19·20 | 5·41 |
| 2·50 | 1·450 | ·0485 | ·850 | 1·53 | 5·77 | ·1210 | 25·90 | 4·89 |
| 3·0 | 1·460 | ·0505 | ·850 | 1·53 | 5 00 | ·1260 | 26·90 | 4·20 |
| 0 | 1·720 | ·0062 | — | — | — | — | — | — |
| 1·50 | 1·795 | ·0218 | ·70 | 1·057 | 4·33 | ·0680 | 11·73 | 4·55 |
| 2·00 | 1·825 | ·0316 | (·62) | 1·110 | 4·05 | (·0652) | 14·70 | (3·28) |
| 3·00 | 1·830 | ·0488 | ·75 | 1·095 | 4·00 | ·1180 | 21·60 | 3·90 |
| 4·00 | 1·900 | .0650 | ·765 | 1·150 | 4·00 | ·170 | 29·00 | 4·24 |
| 5·00 | 1·930 | ·0765 | ·765 | 1·150 | 4·06 | ·190 | 35·30 | 3·79 |

the suffixes denoting the respective values of each quantity for periods $T_{d_1}$ and $T_{d_2}$, $K_1$ and $K_2$ being the circulations for speeds of flow $V_1$ and $V_2$ cm./sec., and the free periods without damping are $[T_0]_1$ and $[T_0]_2$ respectively.

TABLE VI.

Cylinder D = ·970 cm.

| | $d$=5·0 cm. | $d$=4·0 cm. | $d$=4·0 cm. | $d$=3·0 cm. |
|---|---|---|---|---|
| $[x]_{m_1}\frac{\lambda_{d_1}}{T_{d_1}}\cdot\mu_1(T_0)_1^2$ ($\alpha$) | ·160 | ·089 | ·133 | ·103 |
| $[x]_{m_2}\frac{\lambda_{d_2}}{T_{d_2}}\cdot\mu_2(T_0)_2^2$ ($\beta$) | ·116 | ·076 | ·089 | ·083 |
| $V_1$ cm./sec. | 7·00 | 5·00 | 7·54 | 7·70 |
| $V_2$ cm./sec. | 5·29 | 4·30 | 5·00 | 5·04 |
| $\frac{K_1}{K_2}=\frac{\alpha}{\beta}$ | 1·38 | 1·17 | 1·66 | 1·24 |
| $\frac{V_1}{V_2}$ | 1·32 | 1·16 | 1·51 | 1·54 |

Cylinder D = ·64 cm.

| | $d$=5·0 cm. | $d$=4·0 cm. | $d$=3·0 cm. | $d$=2·50 cm. |
|---|---|---|---|---|
| $[x]_{m_1}\frac{\lambda_{d_1}}{T_{d_1}}\cdot\mu_1(T_0)_1^2$ ($\alpha$) | ·0635 | ·0361 | ·0400 | ·0240 |
| $[x]_{m_2}\frac{\lambda_{d_2}}{T_{d_2}}\cdot\mu_2(T_0)_2^2$ ($\beta$) | ·0434 | ·0267 | ·0258 | ·0195 |
| $V_1$ cm./sec. | 4·88 | 3·45 | 4·96 | 3·80 |
| $V_2$ „ | 3·92 | 2·67 | 3·87 | 2·85 |
| $\frac{K_1}{K_2}=\frac{\alpha}{\beta}$ | 1·46 | 1·35 | 1·54 | 1·23 |
| $\frac{V_1}{V_2}$ | 1·25 | 1 30 | 1·28 | 1·33 |

To test the validity of this ratio equality, comparison values obtained are incorporated in Table VI., from which it will be observed good agreement is revealed.

We may furthermore express $[x]_m$ in terms of the maximum velocity $v_m$ of the cylinder in an alternation, for putting

$$v_m = \frac{2\pi}{T_d}[x]_m,$$

we derive the simple relation

$$\frac{v_{m_1}\lambda_{d_1}\mu_1[T_0]_1^2}{v_{m_2}\lambda_{d_2}\mu_2[T_0]_2^2} \doteq \frac{V_1}{V_2},$$

or, generally omitting the suffixes,

$$\frac{v_m}{V} \cdot \lambda_d \mu T_0^2 = \text{a constant for a given } dD.$$

*Deduction of* $\gamma$.

By dimensional consideration, the maximum value of the cross-force given to any cylinder by each eddy should be of the form $\phi(R)\rho d D V^2$, where R = Reynolds's number, and, in the case of simple harmonic considered, $F_0$ is therefore $\phi(R)\rho d D V^2$.

Now from the previous theory $F_0 = \gamma K \rho d V$; therefore, following Karman's assumption that in the formation of detached vortices, a surface of discontinuity is first formed which then breaks up into discrete vortices in which $K = Vl$ ($l$ being the longitudinal spacing of successive vortices in the same row), and accepting the approximate experimental conditions, namely, $l = bD$ generally ($b$ is a constant), we get

$$F_0 = \gamma b \rho d D V^2,$$

and a comparison of both expressions of $F_0$ gives

$$\phi(R) = \gamma b.$$

It is now necessary to express $\phi(R)$ in terms of the drag coefficient $K_d$, and this can be done by adopting a similar treatment to that used by Thom. The total fluid force exerted on the cylinder is

$$F = K_d \rho d D q^2,$$

$q$ being the resultant velocity of the cylinder relative to the fluid.

The component of this force acting across the stream is $\frac{Fv}{q}$ or $K_d \rho d D q v$, and is a maximum when $v = v_m$, whereupon putting $q = (V^2 + v_m^2)^{\frac{1}{2}}$

as an approximation, the maximum component of the cross-force is

$$K_a \rho d D V v_m \left(1 + \frac{v_m^2}{V^2}\right)^{\frac{1}{2}}.$$

Equating this expression to that formerly given leads to the relation,

$$\phi(R) = K_d \frac{v_m}{V}\left(1 + \frac{v_m^2}{V^2}\right)^{\frac{1}{2}},$$

TABLE VII.

$$\gamma = \frac{K_d}{b} \cdot \frac{2\pi}{T_d} \cdot \frac{[x]_m}{V}\left\{1 + \left(\frac{2\pi}{T_d} \cdot \frac{[x]_m}{V}\right)^2\right\}^{\frac{1}{2}}.$$

| Cylinder D. cm. | Period $T_d$. sec. | V. cm./sec. | $[x]_m$. cm. | $K_d$. | $\frac{VD}{\nu}$. | $\gamma$. | $\phi(R)=\gamma b$. |
|---|---|---|---|---|---|---|---|
| ·970 | ·785 | 7·54 | ·715 | ·520 | 640 | ·115 | ·495 |
| „ | 1·09 | 5·00 | ·680 | ·530 | 425 | ·122 | ·525 |
| „ | 1·391 | 4·30 | ·650 | ·600 | 364 | ·114 | ·490 |
| ·64 | 1·360 | 2·74 | ·250 | ·680 | 175 | ·0715 | ·307 |
| „ | 1·035 | 3·45 | ·340 | ·675 | 193 | ·110 | ·474 |
| „ | ·755 | 4·76 | ·390 | ·650 | 217 | ·123 | ·530 |
| 1·317 | 1·460 | 5·00 | ·850 | ·535 | 575 | ·116 | ·498 |
| „ | 1·930 | 4·06 | ·765 | ·600 | 468 | ·101 | ·435 |
| 1·670 | 1·550 | 6·78 | 1·21 | ·525 | 1130 | ·110 | ·474 |

Average value of $\gamma$ from graph on fig. 21 = ·136.

and since $\phi(R) = \gamma b$,

$$\therefore \quad \gamma = \frac{K_d}{b}\frac{v_m}{V}\left(1 + \frac{v_m^2}{V^2}\right)^{\frac{1}{2}}.$$

Finally, substituting for $v_m$ in terms of $[x]_m$ for the purpose of experimental evaluation,

$$\gamma = \frac{K_d}{b} \cdot \frac{2\pi}{T_d} \cdot \frac{[x]_m}{V}\left\{1 + \left(\frac{2\pi}{T_d}\frac{[x]_m}{V}\right)^2\right\}^{\frac{1}{2}}.$$

This equation thus enables $\gamma$ to be deduced in terms of quantities which can be measured experimentally.

A summary of such values is included in Table VII. for the greatest depths of immersions of the cylinders, since it is

reasonable to take these values in order that the theory for a vortex of infinite length, previously assumed, should apply.

Alternately we may obtain $\gamma$ by substituting Karman's value for K in equation (12), thus giving

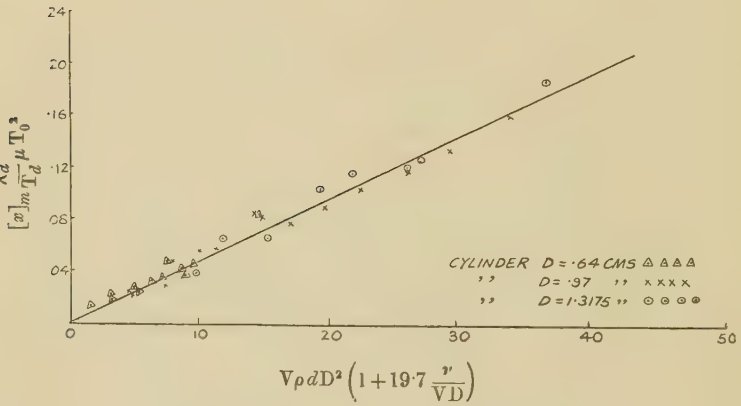
$$[x]_m \frac{\lambda_d}{T_d} \mu T_0^2 = 2{\cdot}52\pi\gamma b V\rho d D^2 \left(1+19{\cdot}7\frac{\nu}{VD}\right) .$$

Verification of such an equality is established by plotting

$$[x]_m \frac{\lambda_d}{T_d} \mu T_0^2 \quad \text{against} \quad V\rho d D^2 \left(1+19{\cdot}7\frac{\nu}{VD}\right),$$

as in fig. 21, where it will be observed that, apart from experimental errors, the average results fall on a straight

Fig. 21.




line. From a knowledge of the slope of this graph, together with $b=4{\cdot}30$ (a value deduced elsewhere)*, the average value of $\gamma$ is ·13, which compares favourably with values given in Table VII. obtained by using the former expression. Moreover, since $\phi(R)=\gamma b$, we can evaluate this coefficient of eddy force, and such values are also included in Table VII., which agree fairly well with Thom's† results.

It is evident that the net circulation around the cylinder producing the cross-force is considerably less than that in a

* Karman and Rubach, *Phys. Zeits.* xiii. p. 49 (1912); Walton, Scientific Proc. Roy. Dublin Soc. xviii. no. 47 (Jan. 1928); Fage, Roy. Aero Soc. Journ. Jan. 1928; Tyler, Phil. Mag., June 1930 and April 1931.

† Thom, R. & M. no. 1373, Dec. 1930.

single detached vortex, and Thom concludes from his results, the side-force is an attribute of the circulation about the cylinder of the same sign as the vortex in formation. This vortex either causes a circulation about the cylinder, or the previously formed vortex on the opposite side of the street, by its withdrawal from the neighbourhood of the cylinder, leaves behind a circulation of the opposite sign.

With either explanation, each vortex must produce sufficient circulation to destroy that left by its predecessor, and then establish sufficient required for the cross-force.

The relative phase detachment of a vortex with respect to a swinging pendulum in water has been examined photographically by Richardson *, and evidence supports the condition that a vortex grows on one side while the pendulum is moving to that side, and is disengaged just after the pendulum begins to swing back.

## Summary.

The free damping of short cylinders and plates attached to the lower end of a vertical reed, and immersed in (*a*) various liquids at rest, and (*b*) in a stream of water is examined.

In the former case, with a fluid of density $\rho$ and viscosity $\eta$, for a cylinder of diameter D cm., under streamline damping,

$$\left(\frac{\lambda_{d_1}}{T_{d_1}} - \frac{\lambda_{d_2}}{T_{d_2}}\right) = \text{a constant}.\, D(d_1 - d_2)\sqrt{\eta\rho},$$

where $\lambda_{d_1}$ and $\lambda_{d_2}$ are the log decs. of the damping for depths of immersion $d_1$ and $d_2$ cm., and $T_{d_1}$ and $T_{d_2}$ the periods of vibration. With a plate of width $b$ cm., inclined at $\theta$ degrees to the plane of vibration,

$$\left(\frac{\lambda_{d_1}}{T_{d_1}} - \frac{\lambda_{d_2}}{T_{d_2}}\right) = \text{a constant}.\, b \sin\theta(d_1 - d_2\sqrt{\eta\rho}.$$

In the latter case, observation of the amplitude of maintained vibration of the cylinders under the influence of the alternate eddies formed at resonance with the system is made, and this, together with a knowledge of

(1) the log dec. of the system without flow at the same depth of immersion,
(2) the restoring force and free period of the system without damping, and
(3) the speed of the fluid,

* Richardson, Roy. Aero. Soc. Journ. March 1927 (proof copy)

enables the net circulation around the cylinders responsible for the cross-force to be determined, the assumption being made that Karman's theory of discontinuity at the surface of a cylinder holds good, in which case the fluid breaks up into discrete eddies.

The drag coefficients for the cylinders are also deduced, and these, together with the fluid speeds and amplitudes of maintained vibration, also permit computation of the cross-forces and circulation around the cylinders. Good agreement is exhibited in both cases, where it is found the net circulation around the cylinders is about ·13 of that in a single detached eddy.

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## CV. *The Graphic Computation of Solar Altitude.*

*By* A. F. Dufton, *M.A., D.I.C.* *

1. In the *Meteorologische Zeitschrift* for August † Dr. Karl Schütte describes a simple graphic method of computing the altitude of the sun. By means of an intersection diagram, comprising a network of three systems of straight lines, he obtains the solution of the equation

$$\sin h = \cos\delta \cos\phi \cos t + \sin\delta \sin\phi, \quad . \quad . \quad (1)$$

which, for latitude $\phi$, expresses the altitude $h$ in terms of the declination $\delta$ and the hour angle $t$.

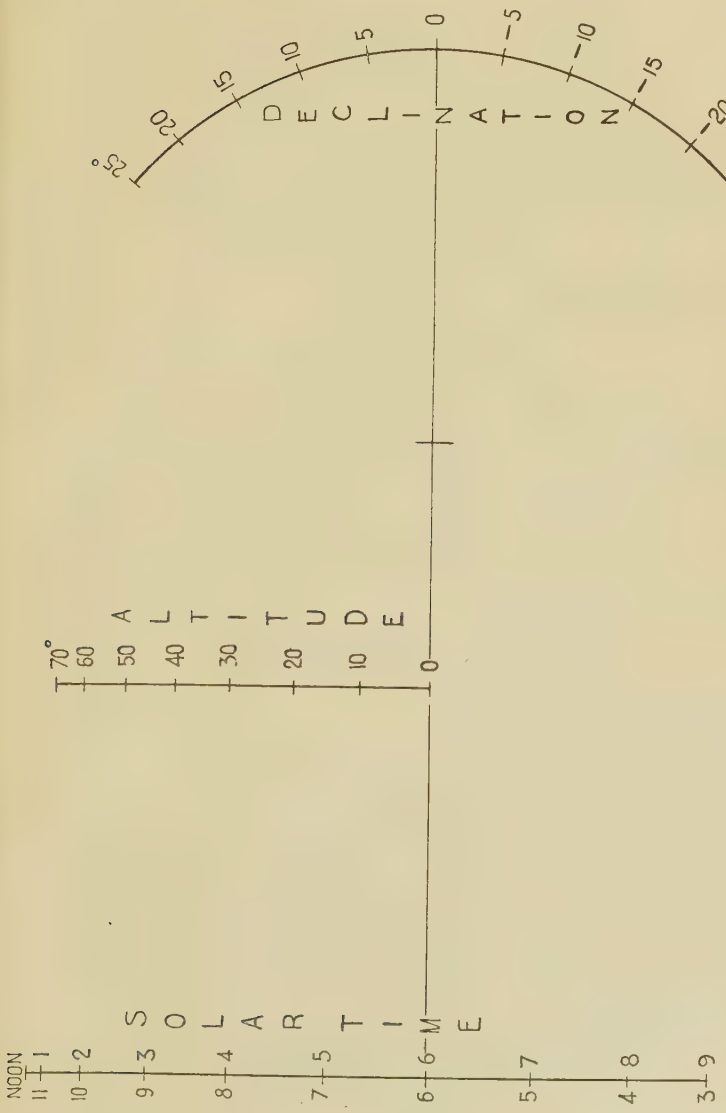
The advantage of an intersection diagram is that it is usually simpler to construct than an alignment diagram. An alignment diagram, however, is generally more convenient to use, as there is no tracing back along a line to read its graduation and interpolation by eye is only necessary upon simple scales.

2. Equation (1) can be expressed in the form

$$\begin{vmatrix} 1 & 0 & -\sin h \\ 0 & 1 & -\cos t \\ 1 & -\cos\phi\cos\delta & -\sin\phi\sin\delta \end{vmatrix} = 0,$$

* Communicated by R. E. Stradling, Esq., Director of Building Research.

† *Meteorologische Zeitsch.* xlviii. (8) p. 314 (1931).




Alignment diagram for solar altitude at latitude 51° 30'.

The figure shows the diagram for the latitude of London (51° 30′ N.). The time-scale is conveniently figured in hours.

which is the condition that the three points,

$$u \qquad -\sin h \qquad =0,$$

$$v \qquad -\cos t \qquad =0,$$

and $$u - v\cos\phi\cos\delta - \sin\phi\sin\delta = 0,$$

expressed in parallel coordinates, $u$ and $v$, are collinear.

Transferred to cartesian coordinates, by comparison with

$$u(1-x/a)+v\,x/a-y/b=0,$$

the first two points are

$$x=0, \qquad y=b\sin h$$

and $$x=a, \qquad y=b\cos t$$

and define two parallel straight scales, while the third point is given by

$$\frac{1-x/a}{1} = \frac{x/a}{-\cos\phi\cos\delta} = \frac{y/b}{\sin\phi\sin\delta}, \quad \cdots \quad (2)$$

the locus being the conic

$$\left(\frac{x}{a\cos\phi}\right)^2 + \left(\frac{y}{b\sin\phi}\right)^2 = \left(\frac{a-x}{a}\right)^2,$$

which reduces to the circle

$$(x-b\cos\phi)^2+y^2=b^2,$$

if $a$ be made equal to $-b\sin\phi\tan\phi$.

3. An alignment diagram, therefore, may be set out by choosing a suitable unit $b$ for the scales of $h$ and $t$ and drawing them at a distance apart equal to $b\sin\phi\tan\phi$. The $\delta$ scale is a circle of radius $b$ with its centre at a distance $b\cos\phi$ from the $h$ scale. The scale of $\delta$ is readily projected, since, from (2),

$$y/x = -(b/a)\tan\delta\tan\phi$$

$$= \tan\delta/\sin\phi.$$

# CVI. *Entropy, Strain, and the Pauli Exclusion Principle.* *By* W. S. Kimball *and* G. Berry *.

## 1. *Newtonian Forces in Momentum Space.*

UNTIL 1931 † physics seems to have neglected the fact that Newtonian forces are operating in velocity and momentum space in addition to and distinct from their operation in ordinary space. It was then pointed out that the equations that determine the classical Maxwell-Boltzmann distribution law are actual equilibrium equations between stress and strain in velocity and momentum space, and that the changes that automatically take place in the *absence* of such equilibrium (as postulated by the Second Law of Thermodynamics) are hence to be explained, interpreted, and accounted for as due to operation of stresses and strains in velocity and momentum space. To be sure, this "explanation" of the Second Law of Thermodynamics would be more complete if formulas had been supplied which gave the actual rates of change in one or more particular illustrations of the Second Law, indicating in each case the controlling part played by the stress and strain in momentum space. In anticipation of this hoped-for result we may note already how forces may act in momentum space.

First, we may be absolutely clear that it is actual Newtonian forces that are operating in momentum space. Imagine a perfect gas of elastic spheres with $n$ molecules per c.c. in a container of constant volume, and disregard gravity. Then for a given temperature the pressure is given by $p=nk\mathrm{T}$. Consider now what happens when heat is added very slowly to this gas. Since the volume is constant and heat is added slowly we may think of each small element of volume as being undisturbed as to the number of molecules it contains, and the gas density hence remains constant and equal to $n$ during the increase of temperature. Each molecular position of equilibrium is occupied, after heating the gas, by a molecule with more energy of agitation than previously. The essential point to be emphasized is that this cannot have been brought about without the operation of Newtonian forces (assuming that it is Newton's laws that determine the molecular behaviour). A molecule at a place $x$ in gas, where $u_0$ is its root mean square agitation velocity,

* Communicated by the Authors.

† W. S. Kimball, "Entropy, Elastic Strain, and the Second Law of Thermodynamics," J. Phys. Chem. xxxv. p. 611 (1931).

cannot emerge later at the same place $x$ with a different $u$ for its agitation velocity without a time rate of change of momentum, involving Newtonian forces. Suppose heat is supplied to it from opposite sides at an equal rate, so that the assumed equilibrium can be maintained. Let the change in momentum in the $x$ direction at the ends of its free path be as indicated in the following table :—

TABLE I.

The $x$-component of Momentum Changes per Average Molecule.

| Impact number. | Momentum change upon even numbered impact. | Momentum between Impacts. | Momentum change upon odd numbered impact. | Impact number. |
|---|---|---|---|---|
| | | $u_0$. | | |
| 2 | $2u_0+3\epsilon=2u_1+\epsilon$ | $-u_1=-(u_0+\epsilon)$ | $-(2u_0+\epsilon)=-(2u_0+\epsilon)$ | 1 |
| | | $u_2=(u_1+\epsilon)$ | $-(2u_2+\epsilon)=-(2u_0+5\epsilon)$ | 3 |
| 4 | $2u_0+7\epsilon=2u_3+\epsilon$ | $-u_3=-(u_2+\epsilon)$ | | |
| | | $u_4=u_3+\epsilon$ | $-(2u_4+\epsilon)=-(2u_0+9\epsilon)$ | 5 |
| 6 | $2u_0+11\epsilon=2u_5+\epsilon$ | $-u_5=-(u_4+\epsilon)$ | | |
| | | $u_6=u_5+\epsilon$ | $-(2u_6+\epsilon)=-(2u_0+13\epsilon)$ | 7 |
| 8 | $2u_0+15\epsilon=2u_7+\epsilon$ | $-u_7=-(u_6+\epsilon)$ | | |
| | | $u_8=u_7+\epsilon$ | $-2u_8=-(2u_0+16\epsilon)$ | 9 |
| 10 | $2u_0+16\epsilon=2u_8$ | $-u_8=-(u_0+8\epsilon)$ | | |

The table represents a very simple transition from the state where $u_0$ is the momentum of the agitation velocity component to a final state where it is $u_8=u_0+8\epsilon$, with the assumption that the latter state of the molecule is reached by eight impacts, each of which adds $\epsilon$ to the last previous momentum magnitude. Thus it will be observed that impacts number 9 and 10 do not change the momentum magnitude, but correspond to the steady state where this momentum is $u_8=u_0+8\epsilon$.

Now the force in ordinary space, due to the change of the vector momentum component of this molecule, is evidently zero, since the odd numbered impacts and even numbered ones have momentum changes of opposite sign, and hence cancel each other ; and corresponding to this we have the familiar fact of Pascal's Law that the hydrostatic pressure is exerted equally in all directions, and hence the vector sum of the forces involved is zero.

In actual detail it will be seen that after the eighth impact, when a steady state is reached, the total negative momentum change (due to the odd numbered impacts) *exceeds* by $8\epsilon$ the

total positive momentum change (from even numbered impacts), or *vice versa*, according as the molecule has had an odd or even number of impacts. Since the time average of this effect is zero, it represents zero resultant force in ordinary space. Thus no resultant force is exerted or work done in ordinary space, corresponding to the transition represented by Table I., when viewed from the long time, large scale point of view.

On the other hand, there is a time rate of change of momentum (and consequent force) involved, whose effect is manifested by the increased momentum between impacts. Since this Newtonian force does no work in ordinary space *except* to increase agitation velocity, it is conveniently referred to as operating only in momentum or velocity space.

Conversely, any change of molecular position in momentum space necessarily involves the operation of forces that may do no resultant work in ordinary space, and hence are to be thought of as operating only in momentum or velocity space.

### 2. *The Geometrical Expression for Weight.*

Boltzmann's famous relation between the entropy, S, of a gas and W, the probability of its state, is given by

$$S = k \log W + C. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Here $k$ is Boltzmann's constant and C is a constant depending on the zero from which entropy is to be measured. The probability W of the gaseous state is calculated rigorously in statistical mechanics by counting the "complexions" or possible arrangements of molecules in phase space. The product of the number of complexions each raised to a suitable power measures the probability, being a physical quantity called (for the sake of definiteness) the weight. This weight represents the W of equation (1) and forms the basis of entropy calculations in statistical mechanics.

There has recently been developed * a "geometrical weight method," based on a physical concept called the "range" or volume occupied per molecule in phase space. The geometrical weight method has been extended † to include the new statistical mechanics of Einstein, Bose, Fermi, and Dirac. Problems usually handled by the classical statistical mechanics all seem within the scope of this new

* W. S. Kimball, "Entropy and Probability," J. Phys. Chem. xxxiii. p. 1558 (1929).

† S. Chandrasekhar, Phil. Mag. ix. p. 621 (1930).

method. One important advantage is that Stirling's approximations and associated difficulties * do not enter at all into this direct geometrical treatment (since factorials never enter). Another advantage is the geometrical picture of weight † which the method affords. This contrasts favourably with counting complexions to find the measure of thermodynamic probability. It is the contention of this and previous papers that the great successes of the statistical method (herein purposely set aside) have been overestimated, and that all of its successes and more in the thermodynamic field can be taken care of by the present mechanical stress and geometrical weight and strain theory.

The most noteworthy achievement of this geometrical weight method is the new explanation which it suggests of the Second Law of Thermodynamics. By using this method it has been shown ‡ that the equilibrium equations which (by the Lagrange method of conditional maxima and minima) determine a state of maximum entropy and probability are equilibrium equations between stress and strain. Entropy then appears as $k$ times the total strain Y :

$$S = k \log W + C = kY. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This suggests that the Second Law of Thermodynamics may be accounted for as being due to the operation of these internal forces or stresses in velocity or momentum as well as ordinary space. The equilibrium state of maximum entropy is thus interpreted as really the state of maximum strain brought about by these forces.

Accordingly, the known equilibrium § equations that lead to the Maxwell-Boltzmann distribution law may be put in the form :

$$-W\frac{f'}{f}du_i = -\lambda m u_i du_i = W\frac{dr_i}{r_i}, \quad . \quad . \quad . \quad (3)$$

wherein the $\lambda$ is an undetermined constant and the $f$ is the Maxwell-Boltzmann distribution function. The $r$'s in the right member are the ranges || or intervals between successsive

* J. Rice, 'Statistical Mechanics for Students,' p. 282.

† Kimball, *loc. cit.* sec. 8.

‡ W. S. Kimball, "Entropy, Elastic Strain, and the Second Law of Thermodynamics," J. Phys. Chem. xxxv. p. 611 (1931), sec. 5.

§ R. C. Tolman, 'Statistical Mechanics,' chap. 4.

|| Kimball, *loc. cit.* sec. 8.

molecules in velocity or momentum space, and the $u$'s are the $x$ components of velocity for each molecule :

$$r_i = u_{i+1} - u_i = \frac{1}{f(u_i)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When $\lambda$ is given its value $-W/kT$ in terms of the absolute temperature T, determined from the distribution function in the usual way *, equations (3) take the form of equilibrium equations between stress and strain † acting in velocity or momentum space :

$$kT = \frac{mu_1 du_1}{dr_1/r_1} = \frac{mu_2 du_2}{dr_2/r_2} = \ldots = \frac{mu_1 du_1}{dr_i/r_i} \quad . \quad . \quad . \quad (5)$$

In the present paper we extend the geometrical weight method to include real gases in which the volume is restricted according to Van der Waals's equation and the energy according to the Pauli exclusion principle ‡. We find that the isothermal bulk modulus for gases obeying these restrictions is the same as that of the classical perfect gas provided molecular attractions are neglected. This holds true both for momentum space and ordinary space. Also the state of maximum entropy or weight for a given internal energy again corresponds to a state of equilibrium between stress and strain acting in momentum space as well as ordinary space, just as was the case for perfect gas. The entropy again appears as $k$ times the total strain in phase space. The velocity distribution function which these equations represent is the same Fermi-Dirac distribution law that is usually derived by the new statistics, and which Chandrasekhar has shown § can be derived by the geometrical weight method.

Perhaps the most important result of the present investigation is the contribution it makes towards a mechanical interpretation of the Pauli exclusion principle. We show that forces acting in momentum space exclude particles from occupied cells with the same vigour that they are excluded from occupied positions in ordinary space, as represented by Van der Waals's constant $b$. Thus Pauli's exclusion principle is explained, generally speaking, as due to the same mechanical tendency as the Second Law of Thermodynamics. Each is a physical aspect of intrinsic processes arising from

* L. B. Loeb, 'Kinetic Theory of Gases,' chap. 4.

† W. S. Kimball, "Entropy, Elastic Strain, and the Second Law of Thermodynamics," J. Phys. Chem. xxxv. p. 611 (1931), sec. 5.

‡ Rice, *loc. cit.* p. 277.

§ Chandrasekhar, *loc. cit.*

objections * which groups of particles have toward compression in phase space. The corrections introduced by the Pauli exclusion principle show the degree of objection which particles feel when compression tends to put additional ones in an occupied phase volume. The mechanical vigour of the objections is the same in momentum space as in ordinary space as measured by the gas law and Van der Waals's equation.

### 3. *The Bulk Modulus and the Stress-strain Relations for Real Gases in Ordinary Space.*

From the gas law we calculate the isothermal bulk modulus by differentiation at constant temperature :

$$p = -\frac{dp}{d\mathrm{V}/\mathrm{V}} = nk\mathrm{T}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Here $p$ is the pressure, V the volume, and $n$ is the number of molecules per c.c. Multiplying both members of equation (5) by $n$ gives the modulus for stress and strain in momentum space :

$$p = \frac{n\,mu\,du}{dr/r} = nk\mathrm{T}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Comparison of (6) and (7) shows the striking fact † that the isothermal bulk modulus $p$ is the same for changes in momentum space as for changes in ordinary space in case of perfect gas.

For real gases obeying Van der Waals's equation

$$\left(p + \frac{a}{\mathrm{V}^2}\right)(\mathrm{V} - b) = \mathrm{RT}$$

the isothermal bulk modulus calculated as above takes a modified form :

$$p' = p + \frac{a}{\mathrm{V}^2} = -\frac{dp'}{d\mathrm{V}/(\mathrm{V}-b)}. \quad . \quad . \quad . \quad (8)$$

If we consider only repulsive forces due to impact the potential energy term $a/\mathrm{V}^2$ due to molecular attractions drops out, and Van der Waals's equation simplifies to

$$p(\mathrm{V} - b) = \mathrm{RT}.$$

* Kimball, *loc. cit.*
† Kimball, *loc. cit.* p. 620.

This simplified form gives, instead of (8),

$$p=-\frac{dp}{dV/(V-b)}=\frac{dp}{\frac{dF}{F}\Big/\left(1-\frac{F}{B}\right)} \quad . \quad . \quad . \quad (9)$$

for the isothermal bulk modulus. In the right member of (9) we let $V=N/F$ and $b/N=1/B$ for the sake of later comparisons (see (14)). Equation (9) shows that if molecular attractions are neglected, and account taken only of molecular size, the modulus of elasticity $p$ is the same as for gases obeying the gas law. Hence the incorporation of what might be called the "Van der Waals exclusion principle" has no effect on the modulus of elasticity.

### 4. *Entropy, Strain, and Weight for Real Gases in Ordinary Space.*

If we use the gas law, and restrict attention to isothermal changes, the First Law of Thermodynamics gives the relation between the entropy change and the work term:

$$dS=\frac{dQ}{T}=\frac{pdV}{T}=R\frac{dV}{V} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The right member shows the relation between entropy and strain.

If we consider real gases represented by Van der Waals's equation, and again restrict attention to isothermal changes, the work term will be modified:

$$dS=\frac{dQ}{T}=\left(p+\frac{a}{V^2}\right)\frac{dV}{T}=R\frac{dV}{(V-b)} \quad . \quad . \quad . \quad (11)$$

Here again the entropy appears as the sum of the strains multiplied by Boltzmann's constant $k$:

$$S=R\Sigma\frac{dV}{V-b}=R\log(V-b)+C=kY. \quad . \quad . \quad (12)$$

It has been shown * for perfect gas in which molecules are mere points how the range in ordinary space, $1/n$, the volume occupied per molecule is related to the weight that measures the thermodynamic probability for one out of N molecules. The N is the number of molecules in a gram molecule, the amount of gas considered

$$W=N(1/n)=V. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

For real gases, following the idea of Van der Waals's equation and his constant $b$, we see that the range, instead

* Kimball, *loc. cit.* p. 1567.

of being $1/n$, will be restricted by the effective volume attributed to each molecule by whatever method is used in calculating $b$. Thus

$$r=\frac{1}{n}-b_1=\frac{1}{n}-\frac{b}{\mathrm{N}}=\frac{1}{\mathrm{F}}-\frac{1}{\mathrm{B}} \quad . \quad . \quad . \quad (14)$$

is the range or volume which each particle has to itself, being the previous range reduced by $b_1$, the effective volume displaced by its physical size. In the right member we let F, a constant distribution function, represent the number of particles per unit volume in ordinary space, just as $f$ represents the number per unit volume in momentum space. Also $\mathrm{B}=1/b_1$ is evidently the number of particles that could be packed into a unit volume in case there is no free motion. Hence the weight per molecule when N are equally likely to be included in the range $r$ is

$$\mathrm{W}=\mathrm{N}r=\frac{\mathrm{N}}{n}-\mathrm{N}b_1=\mathrm{V}-b=\frac{\mathrm{N}}{\mathrm{F}}\left(1-\frac{\mathrm{F}}{\mathrm{B}}\right). \quad . \quad . \quad (15)$$

For N molecules the weight is

$$\mathrm{W}=(\mathrm{V}-b)^{\mathrm{N}}=\left(\frac{\mathrm{N}}{\mathrm{F}}\right)^{\mathrm{N}}\left(1-\frac{\mathrm{F}}{\mathrm{B}}\right)^{\mathrm{N}}, \quad . \quad . \quad . \quad (16)$$

which, when substituted in Boltzmann's equation, gives

$$d\mathrm{S}=\mathrm{N}k\frac{d\mathrm{V}}{\mathrm{V}-b}, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and $$\mathrm{S}=k\log\mathrm{W}+\mathrm{C}=\mathrm{N}k\log(\mathrm{V}-b)+\mathrm{C}=k\mathrm{Y}. \quad . \quad (18)$$

Equations (17) and (18) check equations (11) and (12), and the entropy strain equation (2) shown * to hold for gases obeying the gas law. Here $b$ and B represent the fact that only one particle at a time can occupy a volume displaced by its physical dimensions. We might refer to this as the Van der Waals exclusion principle by analogy with or perhaps a part of the Pauli exclusion principle, which excludes extra particles from an occupied cell in phase space.

### 5. *Entropy, Strain, and Weight for Real Gases in Velocity and Momentum Space.*

As in the classical statistics the number of particles in a unit volume of velocity or momentum space is also represented by a distribution function

$$\frac{d\mathrm{N}}{d\omega}=f(u,\,v,\,w)=n_i=\frac{1}{v}, \quad . \quad . \quad . \quad . \quad (19)$$

* Kimball, *loc. cit.*

where $d\omega = du\,dv\,dw$ is the volume element in the space and $u$, $v$, and $w$ are the velocity components. The volume occupied by a molecule at this place in velocity space is given by the reciprocal of this number as indicated. Where the gas law applies this volume equals the range $(r_i = v_i)$ that leads* to the classical distribution law. It has to be multiplied by N to give the weight per molecule, and the product for N molecules gives the total weight W, being related to entropy and the Maxwell-Boltzmann distribution law in the same familiar way† that the statistical expression for weight is related to them.

For real gases we restrict the range given by (19) according to the Pauli exclusion principle:

$$r_i = \frac{1}{n_i} - a_1 = \frac{1}{f_i} - \frac{1}{\mathrm{A}} = v_i - a_1 = \frac{1}{g_i}. \quad . \quad . \quad (20)$$

This is the free empty range left to a particle at that place in velocity or momentum space after deducting $a_1$, the minimum range it can occupy. Here $\mathrm{A} = \frac{1}{a_1}$ is evidently the number of such minimum ranges or cells in unit volume of momentum space, being the usual symbol‡ used to represent the Pauli exclusion principle. Equation (20) is seen to be identical with (14), except that it applies to velocity or momentum space instead of ordinary space, and suggests possible physical relationship between the Pauli exclusion principle and the Van der Waals exclusion principle.

A causal relationship between these two has already been considered§ and then dismissed‖ by Ehrenfest. The present treatment purports to show that both are aspects of the *modus operandi* of elastic *forces* in a restricted phase space. Meanwhile we note the striking parallelism between $a_1 = 1/\mathrm{A}$ and $b_1 = b/\mathrm{N} = 1/\mathrm{B}$, the latter being the volume in ordinary space that a particle must occupy to the exclusion of other particles by virtue of its very physical existence.

Having obtained the expression for range, we multiply this by N to give the weight per molecule which measures the probability that any one out of N molecules may occupy this particular range:

$$w_i = \mathrm{N}r_i = \frac{\mathrm{N}}{f_i}\left(1 - \frac{f_i}{\mathrm{A}}\right) = \frac{\mathrm{N}}{g_i}; \quad . \quad . \quad . \quad . \quad (21)$$

* Kimball, *loc. cit.*
† Kimball, *loc. cit.*
‡ Chandrasekhar, *loc. cit.*
§ Ehrenfest, 'Nature,' cxix. p. 196 (1927).
‖ Ehrenfest, 'Nature,' cxix. p. 602 (1927).

and for N molecules the weight is

$$W = N^N(r_1 r_2 \ldots r_N) = N^N \left( \ldots \frac{1 - \frac{f_i}{A}}{f_i} \right) = \frac{N^N}{g_1 g_2 \cdots g_N}, \quad (22)$$

where the $g$'s defined by (20) are the reciprocals of the ranges. Substitution in Boltzmann's equation gives the entropy,

$$S = k\left(\sum_1^N \log r_i\right) + C = k\sum_1^N \int \frac{dr_i}{r_i} = kY = Nk \log r + C. \quad (23)$$

Equations (21) and (22) have been given by Chandrasekhar*. They are here shown in terms of the free range (20). It is significant that the weight is still $N^N$ times the product of the free ranges reserved to the particles in momentum space, and the entropy (23) is the sum of the corresponding strains exactly as it was in the case of gas obeying the gas law.

Although the argument of this section refers to velocity or momentum space, it may be generalized † to include phase space (action space), and leads to formulæ of the same form as (21), (22), and (23).

### 6. *The Equilibrium Equations for Stress and Strain in the Fermi-Dirac Statistics.*

To find the maximum value of W subject to a constant energy E we use the method of Lagrange. The function $\phi$ is formed by adding E times an undetermined constant $\lambda$ to W, $\phi = W + \lambda E$. Take the N partial derivatives of the now N independent $u$'s, and set each derivative equal to zero :

$$\frac{\partial \phi}{\partial u_i} = \frac{\partial W}{\partial u_i} + \lambda \frac{\partial E}{\partial u_i} = -W \frac{d}{du_i} \log g_i + \lambda m u_i = 0, \quad (24)$$

or, in view of (20),

$$W \frac{dr_i}{r_i} + \lambda m u_i \, du_i = 0, \ldots\ldots \quad (25)$$

where the partial derivatives are taken with respect to the $x$ components of the molecular velocities. Equations (25) show that there is a constant ratio, namely,

$$-\frac{W}{\lambda} = \frac{m u_1 \, du_1}{dr_1/r_1} = \frac{m u_2 \, du_2}{dr_2/r_2} = \ldots\ldots = -\frac{m u \, du}{\frac{df}{f} \Big/ \left(1 - \frac{f}{A}\right)}, \quad (26)$$

between energy change per molecule (between successive

* Chandrasekhar, *loc. cit.*
† Chanrdasekhar, *loc. cit.*

molecules) and the corresponding strain, just as was shown * to be the case for the Maxwell-Boltzmann distribution. This ratio actually is the elastic modulus per molecule corresponding to the operation of stresses and strains in momentum space, since it has all the characteristics of such a physical constant as shown in detail for the perfect gas case †.

The distribution function corresponding to (24), (25), or (26) takes the form

$$f(u)=\frac{1}{ce^{-\frac{\lambda mu^2}{2W}}+\frac{1}{A}}, \quad . \quad . \quad . \quad . \quad (27)$$

which is the Fermi-Dirac distribution law, thus derived by the geometrical weight method as already shown by Chandrasekhar ‡. It is well known § that the constant $\lambda$ is required by thermodynamics to be $-W/kT$. Substitute this in (26) above, and obtain

$$k\mathrm{T}=\frac{mu\,du}{dr/r}=\frac{mu\,du}{\frac{df}{f}\Big/\left(1-\frac{f}{A}\right)}. \quad . \quad . \quad . \quad (28)$$

Multiplying both members of (28) by $n$ gives

$$p=nk\mathrm{T}=\frac{n\,mu\,du}{dr/r}=-\frac{n\,mu\,du}{\frac{df}{f}\Big/\left(1-\frac{f}{A}\right)}. \quad . \quad . \quad (29)$$

These formulæ are like (6) and (7), and show that in the case of the Fermi-Dirac statistics, as well as the classical law, the elastic modulus per molecule is $k$T. For a unit volume it is equal to the simple pressure formula ($p=nk$T). Likewise the elastic modulus appearing in (29) is the same as that in (9), which refers to real gases in ordinary space. Now (9) represents real gases in which the Van der Waals constant $b$ restricts the volume, and no intermolecular attractions are considered. Thus there appears a striking parallelism (as represented by (28) and (29) on the one hand, and (9) on the other) between the behaviour of molecules in velocity space and ordinary space. In both types of space molecules "object" to being crowded together with the same vigour (same modulus of elasticity) whether their range is curtailed by the Pauli exclusion principle or the Van der Waals exclusion principle. Furthermore, this resistance to

* Kimball, *loc. cit.* p. 617.
† Kimball, *loc. cit.* p. 617.
‡ Chandrasekhar, *loc. cit.*
§ Rice, *loc. cit.* p. 274.

compression is the same in each case as for the simple theory in which the exclusion principles are neglected *.

### 7. *Intermolecular Attractive Forces.*

On the other hand, the elastic modulus given by equation (8) is not the same as in the other cases, since here intermolecular attractions are considered. It is recognized † in connexion with the new statistics that the Fermi-Dirac distribution law, with the negative correction term $\left(\frac{f}{A}\right)$ appearing in (21) and (22), corresponds experimentally to the case of an extra repulsion between particles, and is well verified by a gas of electrons or protons. Where attractive forces exist, as with neutral molecules, the Bose-Einstein statistics ought to hold. If we were to introduce attractive forces into the equilibrium equations like (24) and (25) it seems likely that they would correspond to the new statistical expressions for weight like (21) and (22), except with a positive correction term $\left(+\frac{f}{A}\right)$. Such a change ‡ would give the Bose-Einstein statistics, which can thus probably be fitted into the stress-strain entropy theory. The present treatment, however, considers only the repulsions corresponding to the Pauli exclusion principle.

### 8. *Experimental Verification; Mechanical Interpretation of the Pauli Exclusion Principle.*

The most important verification of the stress-strain equilibrium equations, and the allied entropy-strain equations (2), (18), and (23), is that they suggest a natural and obvious explanation § of the Second Law of Thermodynamics. On the other hand, the ergodic hypothesis has proved quite unsatisfactory ‖ to most physicists.

The present viewpoint is also very suggestive in connexion with the Pauli exclusion principle. Not only is there a striking analogy between the equilibrium equation (9) for ordinary space and (28) and (29) for momentum space, but the modulus of elasticity is the same for the two cases, indicating the operation of the same mechanical causes in the two realms. Even the exclusion of occupied regions in the

* Kimball, *loc. cit.*

† P. Ehrenfest and J. R. Oppenheimer, Phys. Rev. xxxiii. p. 333 (1931).

‡ Chandrasekhar, *loc. cit.*

§ Kimball, *loc. cit.* sec. 7, p. 621.

‖ Tolman, *loc. cit.* p. 39.

two cases introduces the correction terms $\left(\frac{b}{N} = \frac{1}{B}\right)$ for Van der Waals's equation and 1/A for the Pauli exclusion principle in an exactly analogous way. This indicates that their mechanical effect in these force equations is seemingly of the same type.

The operation of the Pauli exclusion principle is thus interpreted as an aspect of the same fundamental process as the Second Law of Thermodynamics. Each is a mechanical aspect of the operation of forces acting in momentum or phase space. The Pauli principle is a kind of measure of the force that excludes particles from occupied cells, and the correction term represents the degree of "objection" which particles feel towards forces tending to put them into already occupied phase positions.

It is significant that the two places where the statistical method seemingly fails to give satisfaction, namely, (*a*) the Second Law of Thermodynamics, and (*b*) the Pauli exclusion principle, are both partly taken care of and interpreted by the present mechanical stress-strain theory. Furthermore, these two fundamental and far-reaching principles are both interpreted as aspects of the same general process, *i. e.*, the operation of stress-strain relations in momentum space.

The dual aspects of light and electricity, as mechanical and corpuscular on the one hand and wave phenomena on the other, are familiar. The present and previous * papers show that certain phenomena also of thermodynamics and statistical mechanics fit nicely when incorporated into the theory of elasticity, and thus can be given a mechanical interpretation as well as one based on probability.

In conclusion, it is to be emphasized that this paper, like the previous one †, does not "extend the meanings of force and strain" ‡, but merely points out that classical Newtonian force and classical zero dimensional strain are actually functioning in momentum space, and that the way they act helps to account for the Second Law of Thermodynamics and the Pauli exclusion principle.

Our thanks are due to Prof. G. E. Uhlenbeck, of Ann Arbor, and Prof. H. A. Kramers, of Utrecht, for very valuable criticism.

Michigan State College,
East Lansing, Michigan.

* Kimball, *loc. cit.*
† Kimball, *loc. cit.*
‡ 'Science Abstracts,' 2744 (August 1931).

CVII. *Operational Calculus.*—Part II.* *The Values of certain Integrals and the Relationships between various Polynomials and Series obtained by Operational Methods.* *By* H. V. Lowry, *Lecturer in Mathematics in the Manchester College of Technology* †.


Summary.

In this part operational methods are used

(*a*) to find the values of certain integrals ;

(*b*) to establish the relationships between Bernouilli polynomials and some other polynomials of a similar type ;

(*c*) to sum some series of Bessel functions of the type

$$\sum_{r=1}^{\infty} r^{-\frac{1}{2}} J_n(2\sqrt{rx}) ;$$

(*d*) to sum series of Bessel functions of the Schlömilch type, such as

$$\sum_{m=1}^{\infty} J_n(mx), \qquad \sum_{m=1}^{\infty} J_n(mx)/m ;$$

(*e*) to sum series of the type

$$\sum_{m=1}^{\infty} J_n(mx) \cos my,$$

which we may call Bessel-Fourier series ;

(*f*) to sum series of Struve functions corresponding to the series of types (*d*) and (*e*).

Of the formulæ proved, those of most interest are :

$$\int_0^x P_m(\cos u) \sin^{m-1}(x-u)du = \frac{\sin^m x}{m}, \quad . \quad . \quad (8)$$

$$\int_0^x \sin^{2n} u \sin^{2n+1}(x-u)du$$

$$= \frac{2n!\ 2n+1!\ 2^{4n-1} \sin^{4n}(\frac{1}{2}x) \cos\frac{1}{2}x}{4n-1!}, \quad . \quad (9)$$

$${}_2\psi_n(x) = {}_1\psi_n(x+\tfrac{1}{2}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$2^n \cdot {}_3\psi_n(x) = {}_2\psi_{n+1}(\tfrac{1}{2}x) - {}_1\psi_{n+1}(\tfrac{1}{2}x), \quad . \quad . \quad (46)$$

$${}_4\psi_n(x) = {}_3\psi_n(x-\tfrac{1}{2}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$


* For Part I. see Phil. Mag. pp. 1033–1048 (May 1932).

† Communicated by the Author.

where the $\psi$'s are polynomials associated with Bernouilli polynomials.

$$x^{-\frac{1}{2}n}\sum_{r=1}^{\infty} r^{-\frac{1}{2}n}\mathrm{J}_n(2\sqrt{rx})$$
$$=\frac{x^{n-1}}{\Pi(n-1)}-\frac{x^n}{2\Pi(n)}-\sum_{r=1}^{\infty}\frac{(-1)^r\mathrm{B}_r x^{n+2r-1}}{2r!\,\Pi(n+2r-1)}. \quad . \quad (51)$$

$$\sum_{r=1}^{\infty}\frac{1}{(\sqrt{a^2+r}+a)^n}$$
$$\sim\frac{1}{2^{n-2}a^{n-2}}-\frac{1}{2^{n+1}a^n}-\sum_{r=1}^{\infty}\frac{(-1)^r\mathrm{B}_r\Pi(n+4r-1)}{2r!\,\Pi(n+2r-1)2^{n+2r}a^{n+4r-1}}. \quad . \quad . \quad . \quad (58)$$

$$x^n 2^{-n+1}\sum_{m=1}^{\infty} m^{-n}\mathrm{J}_n(mx)$$
$$=-\frac{x^{2n}}{\Pi(n)2^{2n}}+\frac{\sqrt{\pi}}{\Pi(n-\frac{1}{2})2^{2n-1}}\left[x^{2n-1}+2\sum_{k=1}^{r}\{x^2-(2k\pi)^2\}^{m-\frac{1}{2}}\right].$$
$$2r\pi<x<2(r+1)\pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

$$2\sum_{m=1}^{\infty} m^{-n}\mathrm{J}_n(mx)\cos ny=-\frac{x^n}{\Pi(n)2^n},\quad \begin{matrix}0<x<y,\\ 0<y<\pi.\end{matrix} \quad (72)$$

$$2x^{\frac{1}{2}}\sum_{m=1}^{\infty} m^{-n-1}\mathrm{S}_n(2m\sqrt{x})$$
$$=-\frac{x^{n+\frac{1}{2}}}{\Pi(n+\frac{1}{2})}+\pi\left[\frac{x^n}{\Pi(n)}+2\sum_{k=1}^{r}\frac{x^n}{\Pi(n)}-\frac{x^n}{\Pi(n-\frac{1}{2})}\int_x^{\infty}\frac{y(x-\frac{1}{4}y^2)^{n-\frac{1}{2}}}{4x^{n-1}}\,dx\right],$$
$$r^2\pi^2<x<(r+1)^2\pi^2. \quad . \quad . \quad . \quad . \quad . \quad (83)$$

$$2x^{\frac{1}{2}n}\sum_{n=1}^{\infty} n^{-n-1}\mathrm{S}_n(2m\sqrt{x})\cos my=-\frac{2x^{n+\frac{1}{2}}}{\sqrt{\pi}\Pi(n+\frac{1}{2})},\quad \begin{matrix}0<x<y,\\ 0<y<\pi.\end{matrix} \quad . \quad . \quad . \quad (86)$$

In the last two equations $\mathrm{S}_m(x)$ is Struve's function of order $m$. In equations (72) and (86) above the results are given for $0<x<y$, but in the text formulæ are given for the sums for all values of $x$.

1. *The Operator for* $\sin^m x$ *and some Integrals obtained from it.*

SINCE

$$\frac{p}{p^2+1} \doteqdot \sin x,$$

$$\sin^2 x \doteqdot \frac{e^{ix}-e^{-ix}}{2i}\,\frac{p}{p^2+1}$$

$$= \frac{1}{2i}\left\{\frac{p-i}{p(p-2i)}\left(\frac{p}{p-i}\right) - \frac{p+1}{p(p+2i)}\left(\frac{p}{p+i}\right)\right\}$$

$$= \frac{1}{2i}\left(\frac{1}{p-2i} - \frac{1}{p+2i}\right)$$

$$= \frac{2}{p^2+2^2}.$$

Similarly,

$$\sin^3 x \doteqdot \frac{3!\,p}{(p^2+1^2)(p^2+3^2)},$$

and so, by induction,

$$\sin^{2n} x \doteqdot \frac{2n!}{(p^2+2^2)(p^2+4^2)\ldots\{p^2+(2n)^2\}}, \quad . \quad . \quad (1)$$

and

$$\sin^{2n+1} x \doteqdot \frac{2n+1!\,p}{(p^2+1^2)(p^2+3^2)\ldots\{p^2+(2n+1)^2\}}. \quad (2)$$

Now, since

$$\mathrm{P}_{2n}(\cos x) \doteqdot \frac{(p^2+1^2)(p^2+3^2)\ldots\{p^2+(2n-1)^2\}}{(p^2+2^2)(p^2+4^2)\ldots\{p^2+(2n)^2\}},* \quad (3)$$

and

$$\mathrm{P}_{2n+1}(\cos x) \doteqdot \frac{(p^2+0^2)(p^2+2^2)\ldots\{p^2+(2n)^2\}}{(p^2+1^2)(p^2+3^2)\ldots\{p^2+(2n+1)^2\}}, \quad (4)$$

we find, by substitution in Borel's integral,

$$\frac{1}{p} f_1(p) f_2(p) \doteqdot \int_0^x h_1(u) h_2(x-u)\,dx, \quad . \quad . \quad (5)$$

* "The Operational Solution of Linear Equations," by Dr. B. v. d. Pol, Phil. Mag. viii. p. 888 (1929).

that

$$\int_0^x P_{2n}(\cos u)\sin^{2n-1}(x-u)du$$

$$\doteqdot \frac{2n-1\,!}{(p^2+2^2)(p^2+4^2)\ldots\{p^2+(2n)^2\}}$$

$$= \frac{\sin^{2n}x}{2n}, \quad \ldots\ldots\ldots\ldots \quad (6)$$

and

$$\int_0^x P_{2n+1}(\cos u)\sin^{2n}(x-u)du$$

$$\doteqdot \frac{2n\,!\,p}{(p^2+1^2)(p^2+3^2)\ldots\{p^2+(2n-1)^2\}}$$

$$= \frac{\sin^{2n+1}x}{2n+1}. \quad \ldots\ldots\ldots\ldots \quad (7)$$

(6) and (7) are both included in

$$\int_0^x P_m(\cos u)\sin^{m-1}(x-u)du = \frac{\sin^m(x)}{m}. \quad (8)$$

In the same way, from (1) and (2) we get

$$\int_0^x \sin^{2n}u\sin^{2n+1}(x-u)du$$

$$\doteqdot \frac{2n\,!\,2n+1\,!\,p}{(p^2+1^2)(p^2+2^2)\ldots\{p^2+(2n)^2\}}$$

$$= \frac{2n\,!\,2n+1\,!\,2^{4n}}{4n\,!}\,p.\sin^{4n}(\tfrac{1}{2}x)$$

$$= \frac{2n\,!\,2n+1\,!\,2^{4n-1}\sin^{4n}(\frac{1}{2}x)\cos(\frac{1}{2}x)}{4n-1\,!}. \quad (9)$$

2. *Properties of Bernouilli Polynomials and the Relationships of these Polynomials to certain other Polynomials.*

From (1) we find that

$$\tfrac{1}{2}\operatorname{cosech}\tfrac{1}{2}p\pi \doteqdot \lim_{n\to\infty} \frac{(n\,!)^2\,2^{2n}}{\pi p(p^2+2^2)\ldots\{p^2-(2n)^2\}}$$

$$\doteqdot \lim_{n\to\infty} \frac{(n\,!)^2\,2^{2n}}{\pi.2n\,!}\int_0^x \sin^{2n}x\,dx.$$

From this limit it is easily shown that, if $x$ is real,

$$\tfrac{1}{2}\operatorname{cosech}\tfrac{1}{2}p\pi \doteqdot r, \quad (r-\tfrac{1}{2})\pi < x < (r+\tfrac{1}{2})\pi\ ^{*}; \quad . \quad (10)$$

so

$$\tfrac{1}{2}\operatorname{cosech}\tfrac{1}{2}p \doteqdot r, \quad (r-\tfrac{1}{2}) < x < (r+\tfrac{1}{2}). \quad . \quad (11)$$

From this operator we easily find that

$$\frac{1}{e^p-1} = \tfrac{1}{2}e^{\frac{1}{2}p}\operatorname{cosech}\tfrac{1}{2}p \doteqdot r, \quad r < x < r+1\ ; \quad . \quad . \quad . \quad (12)$$

$$\frac{e^p}{e^p-1} = \tfrac{1}{2}(1+\coth\tfrac{1}{2}p) \doteqdot r, \quad r-1 < x < r\ ; \quad . \quad . \quad (13)$$

$$\frac{1}{e^p+1} = \frac{e^p}{e^{2p}-1} - \frac{1}{e^{2p}-1} \doteqdot \begin{matrix} 0, & 2r < x < 2r+1, \\ 1, & 2r+1 < x < 2r+2\ ; \end{matrix} \quad (14)$$

and

$$\tfrac{1}{2}\operatorname{sech}\tfrac{1}{2}p = \frac{e^{\frac{1}{2}p}}{e^p+1} \doteqdot \begin{matrix} 0, & 2r-\frac{1}{2} < x < 2r+\frac{1}{2}, \\ 1, & 2r+\frac{1}{2} < x < 2r+\frac{3}{2}. \end{matrix} \quad . \quad . \quad . \quad . \quad (15)$$

Heaviside, followed by other writers, found the functions given by these operators by expanding the operators in powers of $e^{-p}$. In order to be able to do this we must have

$$|\ e^{-p}\ | < 1 \quad \text{that is,} \quad \mathrm{R}(p) > 0,$$

and so we have to operate on $\mathrm{H}(x)$ instead of on 1. This means that, with the operator $\dfrac{1}{e^p-1}$ for example, we get

$$\frac{1}{e^p-1}\,\mathrm{H}(x) \doteqdot 0, \quad x < 0,$$

$$\doteqdot r, \quad r < x < r+1\ ; \ (r \geqq 0).$$

Equation (12) is more general than this because it is true for all real values of $x$.

Similarly, equations (13), (14), (15) are more general than the corresponding equations obtained by expanding in series and operating on $\mathrm{H}(x)$.

Now from equations (11) to (15) we find by successive integration from 0 to $x$ that, when $n$ is a positive integer,

$$\tfrac{1}{2}p^{-n}\operatorname{cosech}\tfrac{1}{2}p \doteqdot \frac{1}{n!}\{(x-\tfrac{1}{2})^n + (x-\tfrac{3}{2})^n + \ \ldots\ + (x-r+\tfrac{1}{2})^n\},$$

$$r-\tfrac{1}{2} < x < r+\tfrac{1}{2}\ ; \quad . \quad . \quad . \quad (16)$$

* See a paper by the author on "Approximation Curves for a Fourier Series," Phil. Mag. x. p. 695 (1930).

$$\frac{p^{-n}}{e^p-1} \doteqdot \frac{1}{n\,!}\{(x-1)^n+(x-2)^n+\ \dots\ +(x-r)^n\},$$

$$r<x<r+1\ ;\quad .\quad .\quad .\quad (17)$$

$$\frac{p^{-n}}{e^p+1} \doteqdot \frac{1}{n\,!}\{(x-1)^n-(x-2)^n+\ \dots\ +(-1)^{r-1}(x-r)^n\},$$

$$r<x<r+1\ ;\quad .\quad .\quad .\quad (18)$$

$$\tfrac{1}{2}p^{-n}\operatorname{sech}\tfrac{1}{2}p \doteqdot \frac{1}{n\,!}\{(x-\tfrac{1}{2})^n+(x-\tfrac{3}{2})^n+\ \dots\ +(x-r+\tfrac{1}{2})^n\},$$

$$r-\tfrac{1}{2}<x<r+\tfrac{1}{2}.\quad .\quad .\quad .\quad (19)$$

From the expansion

$$\frac{1}{e^p-1}=\frac{1}{p}-\frac{1}{2}+\frac{B_1 p}{2\,!}-\frac{B_2 p^3}{4\,!}+\ \dots$$

we find, by partial fractions, that, when $m$ is a positive integer,

$$\frac{p^{-2m}}{e^p-1}=\frac{1}{p^{2m+1}}-\frac{1}{2p^{2m}}+\frac{B_1}{2\,!\,p^{2m-1}}-\ \dots\ +\frac{(-1)^{m-1}B_m}{2m\,!\,p}$$

$$+\sum_{k=-\infty}^{\infty}\frac{1}{(2k\pi i)^{2m}(p-2k\pi i)}$$

$$\doteqdot {}_1\psi_{2m+1}(x)+2(-1)^m\sum_{k=1}^{\infty}\frac{\sin 2k\pi x}{(2k\pi)^{2m+1}},\quad .\quad .\quad .\quad .\quad (20)$$

where

$${}_1\psi_{2m+1}(x)=\frac{x^{2m+1}}{2m+1\,!}-\frac{x^{2m}}{2.2m\,!}+\frac{B_1 x^{2m-1}}{2\,!\,2m-1\,!}-\frac{B_2 x^{2m-3}}{4\,!\,2m-3\,!}$$

$$+\ \dots\ +\frac{(-1)^{m-1}B_m x}{2m\,!}.\quad .\quad .\quad .\quad (21)$$

By differentiating equation (20) we get

$$\frac{p^{-2m+1}}{e^p-1}\doteqdot {}_1\psi_{2m}(x)+2(-1)^m\sum_{k=1}^{\infty}\frac{\cos 2k\pi x}{(2k\pi)^{2m}},\quad .\quad (22)$$

where

$${}_1\psi_{2m}(x)=\frac{d\{{}_1\psi_{2m+1}(x)\}}{dx}$$

$$=\frac{x^{2m}}{2m\,!}-\frac{x^{2m-1}}{2.2m-1\,!}+\ \dots\ +\frac{(-1)^{m-1}B_m}{2m\,!}.\quad .\quad (23)$$

Both (20) and (22) are included in

$$\frac{p^{-n+1}}{e^p-1}\doteqdot {}_1\psi_n(x)+2\sum_{k=1}^{\infty}\frac{\cos\pi(2kx-\frac{1}{2}n)}{(2k\pi)^n},\quad .\quad (24)$$

where

$$ {}_1\psi_n(x) = \frac{x^n}{n!} - \frac{x^{n-1}}{2.n-1!} + \frac{B_1 x^{n-2}}{2!\,n-2!} - \frac{B_2 x^{n-4}}{4!\,n-4!} + \ldots, \quad (25) $$

the polynomial ending in a term in $x$ or a constant.

From equations (17) and (24) it follows that

$$ {}_1\psi_n(x) + 2\sum_{k=1}^{\infty} \frac{\cos \pi(2kx - \frac{1}{2}n)}{(2k\pi)^n} = \frac{1}{n-1!}\{(x-1)^{n-1} + \ldots + (x-r)^{n-1}\}, $$

$$ r < x < r+1. \quad . \quad . \quad (26) $$

The polynomial ${}_1\psi_n(x)$ is closely related to the Bernouilli polynomial of the $n$th order $\phi_n(x)$, in fact

$$ 2m!\,{}_1\psi_{2m}(x) = \phi_{2m}(x) + (-1)^{m-1}B_m, \quad . \quad . \quad (27) $$

$$ 2m+1!\,{}_1\psi_{2m+1}(x) = \phi_{2m+1}(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad (28) $$

but it appears to be more natural to use ${}_1\psi_n(x)$ here, because by using it in preference to $\phi_n(x)$ we can express the relationship between the polynomial and the Fourier series by the one equation (25) instead of the two which are necessary when Bernouilli polynomials are used.

In the same way, starting from the series

$$ \tfrac{1}{2}\operatorname{cosech} \tfrac{1}{2}p = \frac{1}{p} - \frac{B_1(2^2-2)p}{2!\,2^2} + \ldots, $$

we find, by partial fractions, that

$$ \tfrac{1}{2}p^{-n+1}\operatorname{cosech} \tfrac{1}{2}p = {}_2\psi_n(x) + 2\sum_{k=1}^{\infty} \frac{(-1)^k \cos \pi(2kx - \frac{1}{2}n)}{(2k\pi)^n}, $$

$$ . \quad . \quad . \quad (29) $$

where

$$ {}_2\psi_n(x) = \frac{x^n}{n!} - \frac{B_1(2^2-2)x^{n-2}}{2!\,n-2!\,2^2} + \ldots, \quad . \quad . \quad (30) $$

the polynomial ending with a term in $x$ or a constant.

By comparing (29) with (16) we get

$$ {}_2\psi_n(x) + 2\sum_{k=1}^{\infty} \frac{(-1)^k \cos \pi(2kx - \frac{1}{2}n)}{(2k\pi)^n} = \frac{1}{n-1!}\{(x-\tfrac{1}{2})^{n-1} + \ldots + (x-r+\tfrac{1}{2})^{n-1}\}, $$

$$ r - \tfrac{1}{2} < x < r + \tfrac{1}{2}. \quad . \quad . \quad . \quad (31) $$

Starting with the series

$$\frac{1}{e^p+1} = \frac{1}{2} - \frac{B_1(2^2-1)p}{2!} + \frac{B_2(2^4-1)p^3}{4!} \ldots, \quad . \quad (32)$$

and

$$\tfrac{1}{2}\operatorname{sech}\tfrac{1}{2}p = \frac{1}{2}\left\{1 - \frac{E_1 p^2}{2!\,2^2} + \frac{E_2 p^4}{4!\,2^4} - \ldots\right\}, \quad . \quad (33)$$

where $E_1$, $E_2$, ... are Euler's numbers, we find that

$$\frac{p^{-n}}{e^p+1} \doteqdot {}_3\psi_n(x) - 2\sum_{k=1}^{\infty} \frac{\sin \pi\{(2k-1)x - \frac{1}{2}n\}}{\{(2k-1)\pi\}^{n+1}}, \quad (34)$$

and

$$\tfrac{1}{2}p^{-n}\operatorname{sech}\tfrac{1}{2}p \doteqdot {}_4\psi_n(x) + 2\sum_{k=1}^{\infty} \frac{(-1)^k \cos \pi\{(2k-1)x - \frac{1}{2}n\}}{\{(2k-1)\pi\}^{n+1}}.$$

. . . (35)

In these equations, ${}_3\psi_n(x)$ and ${}_4\psi_n(a)$ are the polynomials

$${}_3\psi_n(x) = \frac{x^n}{2.n!} - \frac{B_1(2^2-1)x^{n-1}}{2!\,n-1!} + \frac{B_2(2^4-1)x^{n-3}}{4!\,n-3!} - \ldots,$$

. . . (36)

$${}_4\psi^n(x) = \frac{1}{2}\left\{\frac{x^n}{n!} - \frac{E_1 x^{n-2}}{2!\,n-2!2^2} + \frac{E_2 x^{n-4}}{4!\,n-4!\,2^4} - \ldots\right\}, \quad (37)$$

both polynomials ending with a term in $x$ or a constant.

By comparing equations (34) and (35) with (18) and (19) we get

$${}_3\psi_n(x) - 2\sum_{k=1}^{\infty} \frac{\sin \pi\{(2k-1)x - \frac{1}{2}\}}{\{(2k-1)\pi\}^{n+1}}$$

$$= \frac{1}{n!}\{(x-1)^n - (x-2)^n + \ldots\}, \; r < x < r+1, \quad (38)$$

and

$${}_4\psi_n(x) + 2\sum_{k=1}^{\infty} \frac{(-1)^k \cos \pi\{(2k-1)x - \frac{1}{2}n\}}{\{(2k-1)\pi\}^{n+1}}$$

$$= \frac{1}{n!}\{(x-\tfrac{1}{2})^n - (x-\tfrac{3}{2})^n + \ldots\}, \; r-\tfrac{1}{2} < x < r+\tfrac{1}{2}.$$

. . (39)

From equations (26), (30), (38), and (39) we can find various relationships between the polynomials ${}_1\psi_n(x)$,

${}_2\psi_n(x)$, ${}_3\psi_n(x)$, and ${}_4\psi_n(x)$. Replacing $x$ by $x+1$ in each equation, we get immediately

$$ {}_1\psi_n(x+1) - {}_1\psi_n(x) = \frac{x^{n-1}}{n!}, \quad . \quad . \quad . \quad . \quad (40) $$

$$ {}_2\psi_n(x+1) - {}_2\psi_n(x) = \frac{(x+\frac{1}{2})^{n-1}}{n-1!}, \quad . \quad . \quad (41) $$

$$ {}_3\psi_n(x+1) - {}_3\psi^n(x) = \frac{x^n}{n!}, \quad . \quad . \quad . \quad . \quad (42) $$

$$ {}_4\psi_n(x+1) - {}_4\psi_n(x) = \frac{(x+\frac{1}{2})^n}{n!} \quad . \quad . \quad . \quad (43) $$

Now if we put $x+\frac{1}{2}$ for $x$ in equation (26), it becomes

$$ {}_1\psi_n(x+\tfrac{1}{2}) + 2 \sum_{k=1}^{\infty} \frac{(-1)^k \cos \pi(2kx - \frac{1}{2}n)}{(2k\pi)^n} $$

$$ = \frac{1}{n-1!}\{(x-\tfrac{1}{2})^{n-1} + (x-\tfrac{3}{2})^{n-1} + \ldots + (x-r+\tfrac{1}{2})^{n-1}\}, $$

$$ r - \tfrac{1}{2} < x < r + \tfrac{1}{2}. \quad . \quad . \quad . \quad (44) $$

This equation, with equation (31), shows that

$$ {}_2\psi_n(x) = {}_1\psi_n(x+\tfrac{1}{2}). \quad . \quad . \quad . \quad . \quad . \quad (45) $$

In the same way, from equations (38) and (39)

$$ {}_4\psi_n(x) = {}_3\psi_n(x+\tfrac{1}{2}). \quad . \quad . \quad . \quad . \quad . \quad (46) $$

Further, from equations (26), (31), and (38) we find that

$$ 2^n {}_3\psi_n(x) = {}_2\psi_{n+1}(\tfrac{1}{2}x) - {}_1\psi_{n+1}(\tfrac{1}{2}x); \quad . \quad . \quad (47) $$

alternatively this last equation can be deduced directly from the polynomials by comparing coefficients of the several powers of $x$.

Equations (45), (46), and (47) suffice to express each of the polynomials ${}_2\psi_n(x)$, ${}_3\psi_n(x)$, ${}_4\psi_n(x)$ in terms of ${}_1\psi_n(x)$, and hence in terms of Bernouilli polynomials. (45) gives the expansion of Bernouilli polynomials in powers of $x-\frac{1}{2}$, for by putting $x-\frac{1}{2}$ instead of $x$ we get

$$ {}_1\psi_n(x) = {}_2\psi_n(x-\tfrac{1}{2}). $$

This is the general formula of which the cases $n = 1$ to $n = 6$ are given in Bromwich's 'Infinite Series,' p. 300.

3. *Series of Bessel Functions and Allied Series.*

Since

$$\frac{1}{e^{1/p}-1} = p-\tfrac{1}{2}+\frac{B_1}{2!\,p}-\frac{B_2}{4!\,p^3}+\ \dots,$$

$$\frac{p^{-n}}{e^{1/p}-1} = \frac{1}{p^{n-1}}-\frac{1}{2p^n}+\frac{B_1}{2!\,p^{n+1}}-\frac{B_2}{4!\,p^{n+3}}+\ \dots$$

$$\doteqdot \frac{x^{n-1}}{\Pi(n-1)}-\frac{x^n}{2\Pi(n)}+\frac{B_1x^{n+1}}{2!\,\Pi(n+1)}-\frac{B_2x^{n+2}}{4!\,\Pi(n+3)}+\dots$$

. . . (49)

Also expanding the operator in powers of $e^{-1/p}$, which is legitimate in the contour L but not in the contour C, we get

$$\frac{p^{-n}}{e^{1/p}-1}\mathrm{H}(x) = (p^{-n}e^{-1/p}+p^{-n}e^{-2/p}+\ \dots)\mathrm{H}(x)$$

$$\doteqdot x^{\frac{1}{2}n}\sum_{r=1}^{\infty} r^{-\frac{1}{2}n}\mathrm{J}_n(2\sqrt{rx}),\ x \text{ real} > 0, \quad . \quad . \quad (50)$$

the series on the right-hand side being convergent when $n > \frac{1}{2}$.

Therefore from (49) and (50) we get

$$x^{\frac{1}{2}n}\sum_{r=1}^{\infty} r^{-\frac{1}{2}n}\mathrm{J}_n(\sqrt{rx})$$

$$= \frac{x^{n-1}}{\Pi(n-1)}-\frac{x^n}{2\Pi(n)}-\sum_{r=1}^{\infty}\frac{(-1)^r\mathrm{B}_r x^{n+2r-1}}{2r!\,\Pi(n+2r-1)},$$

$$x \text{ real and} > 0,\ n > \tfrac{1}{2}. \quad . \quad . \quad . \quad (51)$$

In the same way, by expanding each of the operators

$$\frac{p^{-n}e^{1/p}}{e^{2/p}-1},\quad \frac{p^{-n}}{e^{1/p}+1},\quad \frac{p^{-n}e^{1/p}}{e^{2/p}+1}$$

in two different ways, we find that

$$x^{\frac{1}{2}n}\sum_{r=1}^{\infty}(2r-1)^{-\frac{1}{2}n}\mathrm{J}_n\{2\sqrt{(2r-1)x}\}$$

$$= \frac{x^{n-1}}{2\Pi(n-1)}+\sum_{r=1}^{\infty}\frac{(-1)^r\mathrm{B}_r(2^{2r}-2)x^{n+2r-1}}{2r!\,\Pi(n+2r-1)}, \quad . \quad (52)$$

$$x^{\frac{1}{2}n}\sum_{r=1}^{\infty}(-1)^{r+1}r^{-\frac{1}{2}n}\mathrm{J}_n(2\sqrt{rx})$$

$$=\frac{x^n}{2\Pi(n)}-\sum_{r=1}^{\infty}\frac{(-1)^r\mathrm{B}_r(2^{2r}-2)x^{n+2r-1}}{2r\,!\,\Pi(n+2r-1)}, \quad . \quad (53)$$

$$x^{\frac{1}{2}n}\sum_{r=1}^{\infty}(-1)^{r+1}(2r-1)^{-\frac{1}{2}n}\mathrm{J}_n\{2\sqrt{(2r-1)x}\}$$

$$=\frac{x^n}{\Pi(n)}+\sum_{r=1}^{\infty}\frac{(-1)^r\mathrm{E}_r x^{n+2r}}{2r\,!\,\Pi(n+2r)}, \quad . \quad . \quad . \quad . \quad . \quad (54)$$

(52), (53), and (54) being true, when $x$ is real and $>0$ and $n>\frac{1}{2}$.

If we write $x^2/4$ for $x$ in equation (51) it becomes

$$\frac{x^n}{2^n}\sum_{r=1}^{\infty}r^{-\frac{1}{2}n}\mathrm{J}_n(r x)$$

$$=\frac{x^{2n-2}}{2^{2n-2}\Pi(n-1)}-\frac{x^{2n}}{2^{2n+1}\Pi(n)}$$

$$-\sum_{r=1}^{\infty}\frac{(-1)^r\mathrm{B}_r x^{2n+4r-2}}{2^{2n+4r-2}2r\,!\,\Pi(n+2r-1)}. \quad . \quad (55)$$

Now apply Carson's integral to this equation, remembering that

$$\frac{2^n\Pi(n-\frac{1}{2})}{\sqrt{\pi}}\,\frac{p}{(p^2+r)^{n+\frac{1}{2}}}\doteqdot r^{-\frac{1}{2}n}x^n\mathrm{J}_n(\sqrt{r}.x)\,;$$

this gives

$$\frac{\Pi(n-\frac{1}{2})}{\sqrt{\pi}}\sum_{r=1}^{\infty}\frac{1}{(a^2+r)^{n+\frac{1}{2}}}$$

$$=\frac{\Pi(2n-2)}{\Pi(n-1)2^{2n-2}a^{2n-1}}-\frac{\Pi(2n)}{\Pi(n)2^{2n+1}a^{2n}}$$

$$-\sum_{r=1}^{\infty}\frac{(-1)^r\mathrm{B}_r\Pi(2n+4r-2)}{2r\,!\,\Pi(n+2r-1)2^{2n+4r-2}a^{2n+4r-1}}.$$

$$. \quad . \quad . \quad (56)$$

The right-hand side of this equation is not convergent, but it is easy to show that Carson's integral of the sum of the terms after the $r$th in equation (55) is less than the integral of the $r$th term, so that the right-hand side of (56) is an asymptotic series. If we write $a^2=b$ in equation (56) and apply Euler's summation formula to the left-hand side we get equation (56) exactly, thus verifying that this equation is correct.

Further, if we divide equation (55) by $x^n$, and apply Carson's integral again, using the operator

$$\frac{p}{\sqrt{p^2+r}}(\sqrt{p^2+r}-p)^n \doteq r^{\frac{1}{2}n}\mathrm{J}_n(\sqrt{r}.x),$$

we find that

$$\sum_{r=1}^{\infty}\frac{(\sqrt{a^2+r}-a)^n}{r^n\sqrt{a^2+r}} \sim \frac{\Pi(n-2)}{\Pi(n-1)2^{n-2}a^{n-1}} - \frac{\Pi(n)}{\Pi(n)2^{n+1}a^{n+1}} - \sum_{r=1}^{\infty}\frac{(-1)^r\mathrm{B}_r\Pi(n+4-2)}{2r!\,\Pi(n+2r-1)2^{n+4r}a^{n+4r-1}}, \quad (57)$$

in which the left-hand side is convergent when $n > 1$. It does not appear that this asymptotic series can be deduced immediately from Euler's summation formula.

Since

$$\frac{d}{da}(\sqrt{a^2+r}-a)^n = -\frac{(\sqrt{a^2+r}-a)^n}{\sqrt{a^2+r}},$$

it follows that, if we integrate (57) from $a$ to $\infty$, we get

$$\sum_{r=1}^{\infty}\frac{(\sqrt{a^2+r}-a)^n}{r^n} \sim \frac{1}{2^{n-2}a^{n-2}} - \frac{1}{2^{n+1}a^n} - \sum_{r=1}^{\infty}\frac{(-1)^r\mathrm{B}_r\Pi(n+4r-1)}{2r!\,\Pi(n+2r-1)2^{n+2r}a^{n+4r-2}}, \quad (58)$$

the left-hand side being convergent if $n > 2$.

4. *Bessel Function Series of the Schlömilch Type.*

Since

$$\pi\sqrt{p}\coth\pi\sqrt{p} = 1+2\sum_{m=1}^{\infty}\frac{p}{p^2+m^2},$$

$$\pi\sqrt{p}\left\{1+2\sum_{m=1}^{\infty}e^{-2m\sqrt{p}}\right\} = 1+2\sum_{m=1}^{\infty}\frac{p}{p^2+m^2}. \quad (59)$$

Hence, if we operate on 1 we get

$$\sqrt{\pi/x}\left\{1+2\sum_{m=1}^{\infty}e^{-m^2\pi^2/x}\right\} = 1+2\sum_{m=1}^{\infty}e^{-m^2x}, \quad (60)$$

the sides being convergent only if $\mathrm{R}(x) > 0$. (60) is a relationship found by Jacobi*.

* Whittaker and Watson, 'Modern Analysis,' 2nd ed. p. 124, Ex. 18.

Writing $x=1/p$ in (60), we find that

$$\sqrt{\pi p}\left\{1+2\sum_{m=1}^{\infty}e^{-m^2\pi^2 p}\right\}\mathrm{H}(x)=\left\{1+2\sum_{m=1}^{\infty}e^{-m^2/p}\right\}\mathrm{H}(x). \quad . \quad . \quad . \quad (61)$$

If $x$ is real the right-hand side of this equation is

$$\doteqdot 1+2\sum_{m=1}^{\infty}\mathrm{J}_0(2m\sqrt{x}),$$

and the left-hand side is

$$\doteqdot\frac{1}{\sqrt{x}}, \qquad 0<x<\pi^2,$$

and

$$\doteqdot\frac{1}{\sqrt{x}}+\frac{2}{\sqrt{x^2-\pi^2}}+\ldots+\frac{2}{\sqrt{x^2-r^2\pi^2}}, \qquad r^2\pi^2<x<(r+1)^2\pi^2;$$

so

$$1+2\sum_{m=1}^{\infty}\mathrm{J}_0(2m\sqrt{x})=\frac{1}{\sqrt{x}}+2\sum_{k=1}^{r}\frac{1}{\sqrt{x^2-k^2\pi^2}}, \quad r^2\pi^2<x<(r+1)^2\pi^2. \quad (62)$$

If we put $2\sqrt{x}=\mathrm{X}$, equation (62) becomes

$$1+2\sum_{m=1}^{\infty}\mathrm{J}_0(m\mathrm{X})=\frac{2}{\mathrm{X}}+4\sum_{k=1}^{r}\frac{1}{\sqrt{\mathrm{X}^2-2^2k^2\pi^2}}, \qquad 2r\pi<\mathrm{X}<2(r+1)\pi. \quad . \quad . \quad . \quad (63)$$

This equation is a particular case of the type given by Schlömilch; but if we multiply equation (61) by $p^{-n}$, and proceed in the same way as above, we get a series of Bessel functions of the $n$th order, and so generalize equation (63). The final equation, after putting $2\sqrt{x}=\mathrm{X}$, is

$$\frac{\mathrm{X}^{2n}}{2^{2n}\Pi(n)}+\frac{\mathrm{X}^n}{2^{n-1}}\sum_{m=1}^{\infty}m^{-n}\mathrm{J}_n(m\mathrm{X})=\frac{\sqrt{\pi}}{2^{2n-1}\Pi(n-\frac{1}{2})}\left\{\mathrm{X}^{2n-1}\sum_{k=1}^{r}(\mathrm{X}^2-2^2k^2\pi^2)^{n-\frac{1}{2}}\right\}, \qquad 2r\pi<\mathrm{X}<2(r+1)\pi. \quad . \quad . \quad . \quad (64)$$

5. *Bessel-Fourier Series of the Schlömilch Type.*

Since

$$2\sum_{m=1}^{\infty}\frac{p^2q^2}{(p^2+m^2)(q^2+m^2)}$$
$$=2\sum_{m=1}^{\infty}\frac{p^2q^2}{q^2-p^2}\left(\frac{1}{p^2+m^2}-\frac{1}{q^2+m^2}\right)$$
$$=\frac{q^2}{q^2-p^2}\{\pi p\coth\pi p-1\}-\frac{p^2}{q^2-p^2}\{\pi q\coth\pi q-1\}$$
$$=-1+\pi\left\{\frac{pq}{p+q}+\frac{2pq^2}{q^2-p^2}\sum_{k=1}^{\infty}e^{-2k\pi p}\right.$$
$$\left.-\frac{2qp^2}{q^2-p^2}\sum_{k=1}^{\infty}e^{-2k\pi q}\right\},\quad\ldots\quad(65)$$

where $\mathrm{R}(q)>0$, if we write $\sqrt{p}$ for $p$ we get

$$2\sum_{m=1}^{\infty}\frac{pq^2}{(p+m^2)(q^2+m^2)}$$
$$=-1+\pi\left\{\frac{\sqrt{pq}}{\sqrt{p}+q}+\frac{2\sqrt{pq^2}}{q^2-p}\sum_{k=1}^{\infty}e^{-2k\pi\sqrt{p}}\right.$$
$$\left.-\frac{2qp}{q^2-p}\sum_{k=1}^{\infty}e^{-2k\pi q}\right\},\ \mathrm{R}(q)>0.\quad(66)$$

Now in (66) let $p$ and $q$ stand for $x$ and $y$ operators; then, operating first with $q$ on $\mathrm{H}(y)$ (since $\mathrm{R}(q)>0$), we get

$$2\sum_{m=1}^{\infty}\frac{p}{p^2+m^2}\cos my$$
$$=-1+\pi\left\{\sqrt{p}e^{-\sqrt{p}y}+\sqrt{p}(e^{\sqrt{p}y}+e^{-\sqrt{p}y})\sum_{k=1}^{\infty}e^{-2k\pi\sqrt{p}}\right.$$
$$\left.-\sqrt{p}\sum_{k=1}^{\infty}e^{\sqrt{p}(y-2k\pi)}-e^{-\sqrt{p}(y-2k\pi)}\right\},\quad(67)$$

in which $y$ is real and $2r\pi<y<2(r+1)\pi$.

Since there is no restriction on $p$, we can operate with this equation on 1, using the contour C. This gives, after reduction,

$$2\sum_{m=1}^{\infty}e^{-m^2x}\cos my$$
$$=-1+\sqrt{\frac{\pi}{x}}\sum_{k=-\infty}^{k=+\infty}e^{-(y-2k\pi)^2/4x},\quad\ldots\quad(68)$$

which is a series found by Jacobi. Both sides of the equation are convergent only if $R(x) > 0$. This means that, if we change $x$ to $1/p$ in (68), we must operate on $H(x)$. Carrying out this substitution we find that

$$2\sum_{m=1}^{\infty} J_0(2m\sqrt{x})\cos my \doteqdot -1+\left\{\sum_{k=-\infty}^{k=+\infty} e^{-\frac{1}{4}(y-2k\pi)^2 p}\right\}(x^{-\frac{1}{2}}). \quad . \quad . \quad . \quad (69)$$

When $0 < y < \pi$, equation (69) becomes

$$\begin{aligned} 2\sum_{m=1}^{\infty} J_0(2m\sqrt{x})\cos ny &= -1, \qquad 0 < y < y^2/4, \\ &= -1+(x-y^2/4)^{-\frac{1}{2}}, \quad y^2/4 < x < (2\pi-y)^2/4, \\ &= -1+(x-y^2/4)^{-\frac{1}{2}}+\{x-(2\pi-y)^2/4\}^{-\frac{1}{2}}, \\ &\qquad (2\pi-y)^2/4 < x < (2\pi+y)^2/4, \quad . \quad . \quad . \quad (70) \end{aligned}$$

and so on.

Hence, if we write $2\sqrt{x}=X$, we get

$$\begin{aligned} 2\sum_{m=1}^{\infty} J_0(mX)\cos my &= -1, \qquad 0 < X < y, \\ &= -1+2(X^2-y^2)^{-\frac{1}{2}}, \quad y < X < 2\pi-y, \\ &= -1+2(X^2-y^2)^{-\frac{1}{2}}+2\{X^2-(2\pi-y)^2\}^{-\frac{1}{2}}, \\ &\qquad 2\pi-y < X < 2\pi+y, \quad . \quad . \quad . \quad (71) \end{aligned}$$

etc., where X is real and $> 0$, and $0 < y < \pi$.

In the same way if, after putting $x=1/p$, we multiply equation (68) by $p^{-n}$ and then operate on $H(x)$, we find that

$$\begin{aligned} 2\sum_{m=1}^{\infty} m^{-n}J_n(mX)\cos my &= -\frac{X^n}{2^n\Pi(n)}, \qquad 0 < X < y \\ &= -\frac{X^n}{2^n\Pi(n)}+\frac{\sqrt{\pi}}{2^{n-1}\Pi(n-\frac{1}{2})}(X^2-y^2)^{n-\frac{1}{2}}, \\ &\qquad y < X < 2\pi-y, \quad . \quad . \quad . \quad (72) \end{aligned}$$

etc.

From equations (63), (64), and (73) it is easy to derive other identities containing Bessel functions. Thus from equation (63), by using the identity

$$\frac{J_0(rX)}{r^2} = \int_0^X \frac{1}{X}\Big|_0^X \times \frac{dJ_0(rX)}{dX} dX,$$

we get

$$2\sum_{r=1}^{\infty} \frac{J_0(rX)}{r^2} = 2X - \frac{X^2}{4}, \qquad 0 < X < 2\pi$$

$$= 2X - \frac{X^2}{4} + \frac{1}{4}(X^2 - 4\pi^2)^{\frac{1}{2}} - 8\pi \cos^{-1}\left(\frac{2\pi}{X}\right),$$

$$2\pi < X < 4\pi. \quad . \quad . \quad . \quad (73)$$

Also, by integrating equation (71) from 0 to $y$, we get

$$2\sum_{r=1}^{\infty} \frac{J_0(rX)\sin ry}{r}$$

$$= -y, \qquad 0 < X < y$$

$$= -y + 2\sin^{-1}\frac{y}{X}, \qquad y < X < 2\pi - y$$

$$= -y + 2\sin^{-1}\frac{y}{X} - 2\sin^{-1}\frac{2\pi - y}{X} + 2\sin^{-1}\frac{2\pi}{X},$$

$$2\pi - y < X < 2\pi + y, \quad . \quad . \qquad (74)$$

etc.

From (74), by another integration,

$$2\sum_{r=1}^{\infty} \frac{J_0(rX)\cos ry}{r^2} - 2\sum_{r=1}^{\infty} \frac{J_0(rX)}{r^2}$$

$$= \frac{y^2}{2}, \qquad 0 < X < y,$$

$$= \frac{y^2}{2} - 2y\sin^{-1}\frac{y}{X} - 2(X^2 - y^2)^{\frac{1}{2}} + 2X,$$

$$y < X < 2\pi - y, \quad . \quad . \quad . \quad (75)$$

etc.

By combining equations (75) and (73) we find that

$$2\sum_{r=1}^{\infty} \frac{J_0(rX)\cos ry}{r^2}$$

$$= \frac{y^2}{2} + 2X - \frac{X^2}{4}, \qquad 0 < X < y,$$

$$= \frac{y^2}{2} + 2y\sin^{-1}\frac{y}{X} - 2(X^2 - y^2)^{\frac{1}{2}} + 4X - \frac{X^2}{4},$$

$$y < X < 2\pi - y\,; \quad . \quad . \quad . \quad (76)$$

in forming these equations it should be noticed that the interval $2\pi - y$ to $2\pi + y$ in equation (76) overlaps two intervals in equation (73).

### 6. *Series containing Struve's Functions.*

All the work in Section 5 can be repeated with Struve's functions $S_n(x)$, where $S_n(x)$ is defined by

$$S_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}x)^{n+2k+1}}{\Pi(n+k+\frac{1}{2})\,\Pi(k+\frac{1}{2})}.$$

We proceed as follows. From equation (59)

$$\frac{1}{\sqrt{p}} + 2\sum_{m=1}^{\infty} \frac{\sqrt{p}}{p+m^2} = \pi \left\{ 1 + 2\sum_{m=1}^{\infty} e^{-2m\pi\sqrt{p}} \right\}. \quad (77)$$

Operating with this equation on 1 we get

$$2\sqrt{\frac{x}{\pi}} + 2\sum_{m=1}^{\infty} e^{-m^2x} \frac{\operatorname{erf}(-mi\sqrt{x})}{mi}$$

$$= \pi + 2\pi \sum_{m=1}^{\infty} \left\{ 1 - \operatorname{erf}\left(\frac{m\pi}{\sqrt{x}}\right) \right\}, \quad (78)$$

because

$$\frac{\sqrt{p}}{p+m^2} = \frac{1}{\sqrt{p}} \frac{p}{p+n^2} = \frac{1}{\sqrt{p}} e^{-n^2x}$$

$$= e^{-n^2x} \frac{1}{\sqrt{p-n^2}}$$

$$\doteqdot \frac{e^{-n^2x}}{\sqrt{\pi}} \left\{ \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{n^2x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{n^4x^{\frac{5}{2}}}{\frac{5}{2}\cdot 2!} + \dots \right\}$$

$$= \frac{e^{-n^2x}}{\sqrt{\pi}} \int_0^x x^{-\frac{1}{2}} e^{-n^2x}\, dx$$

$$= \frac{e^{-n^2x}}{\sqrt{\pi}} \int_0^{ni\sqrt{x}} \frac{2e^{-u^2}}{ni}\, du$$

$$= \frac{e^{-n^2x} \operatorname{erf}(ni\sqrt{x})}{ni}, \quad \dots\dots \quad (79)$$

where

$$\operatorname{erf} t = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\, du.$$

Putting $x=1/p$ in equation (78), and multiplying by $p^{-n}$, we get

$$\left[\frac{2}{p^{n+\frac{1}{2}}}+2\sum_{m=1}^{\infty}\frac{p^{-n}\,e^{-m^2/p}\,\mathrm{erf}(mi/\sqrt{p})}{ni}\right]\mathrm{H}(x)$$
$$=\pi p^{-n}\left[1+2\sum_{m=1}^{\infty}\{1-\mathrm{erf}(m\pi\sqrt{p})\}\right]\mathrm{H}(x). \quad (80)$$

Now

$$\left\{\frac{p^{-n}\,e^{-m^2/p}\,\mathrm{erf}(ni/\sqrt{p})}{ni}\right\}\mathrm{H}(x)$$
$$\doteq\frac{2}{\sqrt{\pi}}\left\{\frac{x^{n+\frac{1}{2}}}{\Pi(n+\frac{1}{2})}+\cdots\right.$$
$$\left.+\frac{(-1)^r\,\Pi(\frac{1}{2})\,x^{n+r+\frac{1}{2}}\,m^{2r}}{\Pi(n+r+\frac{1}{2})\,\Pi(r+\frac{1}{2})}+\cdots\right\}=\frac{x^{\frac{1}{2}n}\,\mathrm{S}_n(2m\sqrt{x})}{m^{n+1}}.$$
$$\ldots \quad (81)$$

(81) is true if $x$ is real and positive. If $x$ is negative it can be shown by Jordan's Lemma that its value is zero. Further

$$[p^{-n}\{1-\mathrm{erf}(m\pi\sqrt{p})\}]\,\mathrm{H}(x)$$
$$=\left[p^{-n}-m\pi\,p^{-n+\frac{1}{2}}+\frac{m^3\pi^3}{3}p^{-n+\frac{3}{2}}-\cdots\right]\mathrm{H}(x)$$
$$=x^n\left[\frac{1}{\Pi(n)}-\frac{1}{\Pi(n-\frac{1}{2})}\left\{\frac{m\pi}{\sqrt{x}}-\frac{(m-\frac{1}{2})}{3}\left(\frac{m\pi}{\sqrt{x}}\right)^3+\cdots\right\}\right]$$
$$=\frac{x^n}{\Pi(n)}-\frac{x^n}{\Pi(n-\frac{1}{2})}\int_x^{\infty}\frac{m\pi(x-m^2\pi^2)^{n-\frac{1}{2}}}{2x^{n+1}}dx. \quad \ldots \quad (82)$$

Clearly this equation is true only if $x>m^2\pi^2$, so that the infinite series is convergent. When $x<m^2\pi^2$

$$p^{-n}\{1-\mathrm{erf}\,(m\pi\sqrt{p})\}\mathrm{H}(x)$$
$$=p^{-n}\left\{1-\frac{2}{\sqrt{\pi}}\int_0^{m\pi\sqrt{p}}e^{-u^2}du\right\}\mathrm{H}(x)$$
$$=\left[\frac{2p^{-n}}{\sqrt{\pi}}\int_{m\pi\sqrt{p}}^{\infty}e^{-u^2}du\right]\mathrm{H}(x)$$
$$=\left[\frac{p^{-n}e^{-m^2\pi^2p}}{\sqrt{\pi}}\int_0^{\infty}\frac{e^{-\lambda}d\lambda}{\sqrt{\lambda+m^2\pi^2p}}\right]\mathrm{H}(x)$$
$$=e^{-m^2\pi^2p}f(p).\mathrm{H}(x),$$

where $f(p)$ is such that $\left|\frac{f(p)}{p}\right| \to 0$, as $|p| \to \infty$, when $m > -3/2$.

This result shows that

$$p^{-n}\{1-\operatorname{erf}(m\pi\sqrt{p})\}\mathrm{H}(x) = \frac{1}{2\pi i}\int_{\mathrm{L}} e^{p(x-m^2\pi^2)}\frac{f(p)}{p}\,dp,$$

which can be shown by Jordan's Lemma to be zero if $x < m^2\pi^2$. Thus we have shown that when $x < m^2\pi^2$ the right-hand side of equation (82) is zero.

Using equations (81) and (82), equation (82) becomes

$$\frac{x^{n+\frac{1}{2}}}{\Pi(n+\tfrac{1}{2})} + 2x^{\frac{1}{2}n}\sum_{m=1}^{\infty}\frac{\mathrm{S}_n(2m\sqrt{x})}{m^{n+1}} = \pi\left[\frac{x^n}{\Pi(n)} + 2\sum_{k=1}^{r}\left\{\frac{x^n}{\Pi(n)} - \frac{x^n}{\Pi(n-\frac{1}{2})} \int_x^{\infty} \frac{k\pi(x-k^2\pi^2)^{n-\frac{1}{2}}}{2x^{n+2}}dx\right\}\right],$$

$$r^2\pi^2 < x < (r+1)^2\pi^2. \quad . \quad . \quad . \quad (83)$$

In the particular case $n=0$ equation (83) takes the form

$$2\sqrt{\frac{x}{\pi}} + 2\sqrt{x}\sum_{m=1}^{\infty}\frac{\mathrm{S}_0(2m\sqrt{x})}{m} = \pi\left[1+2\sum_{k=1}^{r}\left\{1-\frac{1}{\sqrt{\pi}}\int_x^{\infty}\frac{k\pi(x-k^2\pi^2)^{-\frac{1}{2}}}{2x}\,dx\right\}\right]$$

$$= \pi\left[1+2\sum_{k=1}^{r}\left\{1-\frac{1}{\sqrt{\pi}}\sin^{-1}\left(\frac{k\pi}{\sqrt{x}}\right)\right\}\right],$$

$$r^2\pi^2 < x < (r+1)^2\pi^2. \quad . \quad . \quad . \quad (84)$$

If $2\sqrt{x} = \mathrm{X}$, (84) becomes

$$2\sum_{m=1}^{\infty}\frac{\mathrm{S}_0(m\mathrm{X})}{m} = -\frac{2}{\sqrt{\pi}} + \frac{\pi}{\mathrm{X}}\left[1+2\sum_{k=1}^{r}\left\{1-\frac{1}{\sqrt{\pi}}\sin^{-1}\left(\frac{2k\pi}{\mathrm{X}}\right)\right\}\right],$$

$$2r\pi < \mathrm{X} < 2(r+1)\pi. \quad . \quad . \quad . \quad (85)$$

In the same way we can find Struve-Fourier series

corresponding to the Bessel-Fourier series in equation (70). In particular, if $0 < y < \pi$,

$$2x^{\frac{1}{2}n} \sum_{m=1}^{\infty} \frac{S_n(2m\sqrt{x}) \cos my}{m^{n+1}}$$

$$= -\frac{2x^{n+\frac{1}{2}}}{\sqrt{\pi}\Pi(n+\frac{1}{2})}, \qquad 0 < x < y,$$

$$= -\frac{2x^{n+\frac{1}{2}}}{\sqrt{\pi}\Pi(n+\frac{1}{2})} + \frac{\pi x^n}{\Pi(n)}$$

$$\left\{1 - \frac{\Pi(n-\frac{1}{2})}{\Pi(n)} \int_x^{\infty} \frac{y\left(x-\frac{y^2}{4}\right)^{n-\frac{1}{2}}}{4x^{n+1}}\, dx\right\},$$

$$y < x < 2\pi - y \quad . \quad . \quad . \quad (86)$$

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CVIII. *The Hume-Rothery Relationship between the Ionization Potentials of the Elements and their Atomic Number.* *By* H. Yagoda, *B.Sc.* (*Chem.Eng.*), *Sc.M.*, *Sea Gate, New York, U.S.A.**

HUME-ROTHERY has demonstrated that the interatomic distance between the atoms in a crystal ($d$) is connected with the atomic number of the element (Z), and the total quantum number of its outermost shell of electrons ($n$) by the simple relationship [1] :—

$$\frac{d}{n} = \left(\frac{1}{\alpha Z}\right)^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this equation $\alpha$ and $x$ are constants for the members of any one subgroup of the periodic table, provided that the coordination number of the atoms remains constant. The values of $x$ vary slightly from group to group, but can be considered as approximating the fraction 1/3 for the initial groups of the periodic system. From this primary relationship, Hume-Rothery has deduced that $n^2V$ is, approximately, a linear function of $Z^{2/3}$, where V represents the ionization potential of the atom [2]. On further analysis of this connexion between $n^2V$ and $Z^{2/3}$, I have found that only in the case of the neutral alkali atoms is a variation with $Z^{2/3}$ warranted, and that, in general, a functional relationship of the form

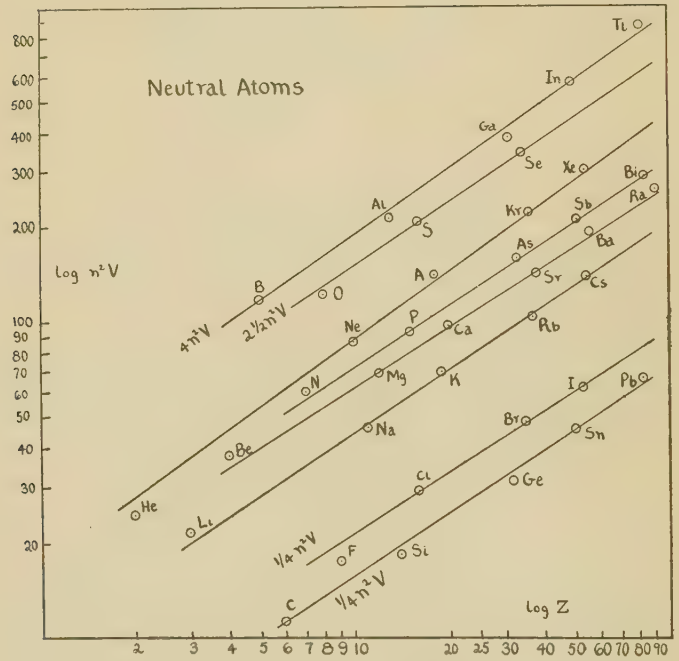
$$\log n^2V = a \log Z + b \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* Communicated by the Author.

represents the ionization potentials of the atoms with greater accuracy.

The lines in figures 1 and 2 are the loci of the equations of form (2) shown in Tables I.–VIII. In order to avoid the crowding together of the lines in the figures, multiples of $n^2V$ were plotted for several of the groups. Whenever the available data on the ionization potentials permitted, the

Fig. 1.

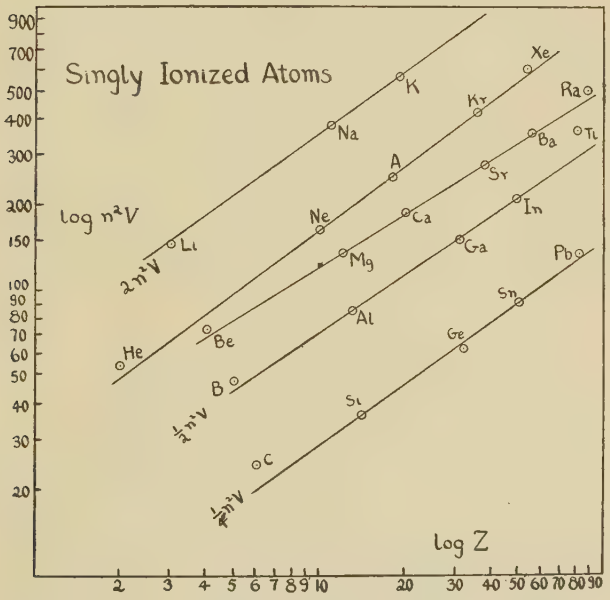



Graph showing the relationship between log $n^2V$ and log Z for the neutral atoms.

coefficients of the equations were evaluated by averaging the coordinates of only the inner elements of each group. The properties of the elements belonging to the initial and seventh periods have, therefore, not influenced the magnitude of the coefficients in the equation for the group. For many elements, the ionization potentials cannot be calculated from spectroscopic data, as their spectra have not been analyzed into terms. However, several rough estimates of this

constant are given in the literature. Evidently these estimates cannot be employed for testing the precision of the Hume-Rothery relationship, but by assuming the validity of the $Z^{2/3}$ variation of $n^2V$, then the probable value of the ionization potential can be selected from the several estimates available, by interpolating the group equation. This has been done for the neutral atoms of the 6th, 7th, and 8th groups, and also for all the singly ionized atoms studied, excepting the noble gases and the alkaline earths. The

Fig. 2.




Graph showing the relationship between log $n^2V$ and log Z for the singly ionized atoms.

points on the graphs, therefore, represent either the ionization potential directly available from the spectrum of the atom, or else the particular value, of the several described in the literature, which agrees best with the present theorem. The ionization potentials represented in Tables I.–VIII. were obtained chiefly from the International Critical Tables[3] and the Landolt-Börnstein "Tabellen"[4]. The data for the singly ionized atoms were invariably taken from the latter work. In the few cases where the published estimates are

very discordant, the particular value chosen will be indicated subsequently.

It can be seen from fig. 1, in which log $n^2V$ has been plotted against log Z, that the respective lines for the eight

Table I.

Ionization Potentials of the Noble Gases.

I. Log $n^2V=0{\cdot}7200$ log $Z+1{\cdot}2303$.
II. Log $n^2V=0{\cdot}7671$ log $Z+1{\cdot}4404$.

| Element. | Z. | *n*. | Neutral atom. | | Singly ionized atom. | |
|---|---|---|---|---|---|---|
| | | | Obs. | Calc. I. | Obs. | Calc. II. |
| Helium ..... | 2 | 1 | 24·48 | 28·0 | 54·16 | 46·9 |
| Neon ..... | 10 | 2 | 21·5 | 22·3 | 40·77 | 40·3 |
| Argon..... | 18 | 3 | 15·69 | 15·1 | 27·82 | 28·1 |
| Krypton..... | 36 | 4 | 13·94 | 14·0 | 26·4 | 26·7 |
| Xenon..... | 54 | 5 | 12·08 | 12·0 | (24) | 23·6 |
| Radon..... | 86 | 6 | — | 11·6 | — | 23·3 |

Table II.

Ionization Potentials of the Alkalies.

I. Log $n^2V=0{\cdot}6664$ log $Z+0{\cdot}9783$.
II. Log $n^2V=0{\cdot}7438$ log $Z+1{\cdot}5041$.

| Element. | Z. | *n*. | Neutral atom. | | *n*. | Singly ionized atom. | |
|---|---|---|---|---|---|---|---|
| | | | Obs. | Calc. I. | | Obs. | Calc. II. |
| Lithium ..... | 3 | 2 | 5·37 | 4·94 | 1 | 75·28 | 72·3 |
| Sodium ..... | 11 | 3 | 5·12 | 5·23 | 2 | 47·5 | — |
| Potassium ..... | 19 | 4 | 4·32 | 4·22 | 3 | 31·7 | — |
| Rubidium ..... | 37 | 5 | 4·16 | 4·19 | 4 | (16) | 29·3 |
| Cæsium ..... | 55 | 6 | 3·88 | 3·82 | 5 | — | 25·2 |
| Eka-cæsium..... | 87 | 7 | — | 3·81 | 6 | — | 24·4 |

periodic groups are roughly parallel to each other ; meaning that $n^2V$ is a linear function of $Z^a$, where the exponent $a$ has approximately the same value throughout the periodic system. From the equations (I.) of each table it can be seen that the values of $a$ averages close to 2/3, but that the departure from this fractional value is often quite marked,

and that only in the case of the alkalies has the coefficient of log Z a magnitude (0·6664) which can be accurately represented by the fraction 2/3. In the 4th group the coefficient is also close to 2/3, viz., 0·6669, but the equation was fitted only to carbon and tin, and so does not represent an average value for the group.

TABLE III.

Ionization Potentials of the Alkaline Earths.

I. Log $n^2V=0{\cdot}6316 \log Z+1{\cdot}1605$.
II. Log $n^2V=0{\cdot}6188 \log Z+1{\cdot}4667$.

| Element. | Z. | *n*. | Neutral atom. | | Singly ionized atom. | |
|---|---|---|---|---|---|---|
| | | | Obs. | Calc. I. | Obs. | Calc. II. |
| Beryllium ..... | 4 | 2 | 9·28 | 8·69 | 18·14 | 17·3 |
| Magnesium ..... | 12 | 3 | 7·61 | 7·72 | 14·97 | 15·1 |
| Calcium ..... | 20 | 4 | 6·09 | 6·02 | 11·82 | 11·7 |
| Strontium ..... | 38 | 5 | 5·67 | 5·78 | 10·98 | 11·1 |
| Barium ..... | 56 | 6 | 5·19 | 5·09 | 9·95 | 9·79 |
| Radium ..... | 88 | 7 | (5·4) | 5·01 | 10·23 | 9·53 |

TABLE IV.

Ionization Potentials of 3rd Group Atoms.

I. Log $n^2V=0{\cdot}6413 \log Z+1{\cdot}0745$.
II. Log $n^2V=0{\cdot}6870 \log Z+1{\cdot}4620$.

| Element. | Z. | *n*. | Neutral atom. | | Singly ionized atom. | |
|---|---|---|---|---|---|---|
| | | | Obs. | Calc. I. | Obs. | Calc. II. |
| Boron ..... | 5 | 2 | 8·33 | — | 23·98 | 21·9 |
| Aluminium ..... | 13 | 3 | 5·96 | 6·84 | 18·75 | — |
| Gallium ..... | 31 | 4 | 5·97 | 6·72 | 18·9 | 19·2 |
| Indium ..... | 49 | 5 | 5·76 | — | (16·8) | — |
| Thallium ..... | 81 | 6 | 6·08 | 5·49 | 20·30 | 16·5 |

Figures 1 and 2 also show that in nearly all cases the first member of each group exhibits definite departures from the linear relationship. This anomalous behaviour of the elements of the first period is characteristic of most of the group relationships in the periodic system. It also appears, judging from the ionization potential of the radium atom

(Table III.), that another deviation from the linear relationship occurs in the seventh period. Here the exceptional character of the radium atom (apart from its nuclear instability) is quite apparent. For the ionization potentials of all the alkaline earth atoms diminish progressively as the group is descended from beryllium to barium, but the

TABLE V.

Ionization Potentials of 4th Group Atoms.

I. Log $n^2V=0{\cdot}6669 \log Z+1{\cdot}1323$.
II. Log $n^2V=0{\cdot}7134 \log Z+1{\cdot}3480$.

| Element. | Z. | *n*. | Neutral atom. | | Singly ionized atom. | |
|---|---|---|---|---|---|---|
| | | | Obs. | Calc. I. | Obs. | Calc. II. |
| Carbon..... | 6 | 2 | 11·22 | — | 24·28 | 20·0 |
| Silicon ..... | 14 | 3 | 8·12 | 8·77 | 16·27 | — |
| Germanium..... | 32 | 4 | 7·85 | 8·56 | 15·6 | 16·4 |
| Tin ..... | 50 | 5 | 7·37 | — | 14·52 | — |
| Lead..... | 82 | 6 | 7·38 | 7·13 | 14·97 | 14·3 |

TABLE VI.

Ionization Potentials of 5th Group Atoms.

I. Log $n^2V=0{\cdot}6577 \log Z+1{\cdot}1994$.

| Element. | Z. | *n*. | Neutral atom. | | Singly ionized atom. | |
|---|---|---|---|---|---|---|
| | | | Obs. | Calc. I. | Obs. | Calc. II. |
| Nitrogen ..... | 7 | 2 | 14·48 | 14·2 | 29·47 | — |
| Phosphorus ... | 15 | 3 | 10·3 | 10·5 | 19·81 | — |
| Arsenic ..... | 33 | 4 | 9·96 | 9·81 | — | — |
| Antimony ..... | 51 | 5 | 8·35 | 8·38 | — | — |
| Bismuth ..... | 83 | 6 | 8·0 | 8·00 | 29·5 | — |

ionization potential of the radium atom is greater than that of barium. This is also characteristic of certain elements of the 6th period, as lead and thallium, which are the last members of their respective groups. Comparison of the calculated with the observed values of the ionization potential shows that though, in these cases, the equations do not predict a reversal in the order of progression of the constant,

the calculated values are very close to that of the homologous element directly above it.

The ionization potential of the neutral arsenic atom is given as 9·4 volts in the 'International Critical Tables,' and 9·96 volts in the Landolt-Börnstein "Tabellen." The value of

TABLE VII.

Ionization Potentials of the 6th Group Atoms.

I. Log $n^2V=0·6518 \log Z+1·1836$.

| Element. | Z. | *n*. | Neutral atom. | | Singly ionized atom. | |
|---|---|---|---|---|---|---|
| | | | Obs. | Calc. I. | Obs. | Calc. II. |
| Oxygen..... | 8 | 2 | 13·56 | 14·8 | 34·93 | — |
| Sulphur ..... | 16 | 3 | 10·31 | — | 23·30 | — |
| Selenium ..... | 34 | 4 | (9·5) | — | — | — |
| Tellurium ..... | 52 | 5 | — | $8·0_3$ | — | — |
| Polonium..... | 84 | 6 | — | 7·6 | — | — |

TABLE VIII.

Ionization Potentials of the 7th Group Atoms.

I. Log $n^2V=0·7121 \log Z+1·1904$.

| Element. | Z. | *n*. | Neutral atom. | | Singly ionized atom. | |
|---|---|---|---|---|---|---|
| | | | Obs. | Calc. I. | Obs. | Calc. II. |
| Fluorine ..... | 9 | 2 | 17·4 | 18·5 | 34·5 | — |
| Chlorine ... ..... | 17 | 3 | 12·96 | — | 22·5 | — |
| Bromine ..... | 35 | 4 | 12·2 | — | — | — |
| Iodine ..... | 53 | 5 | 10 | 10·5 | — | — |
| Eka-iodine ..... | 85 | 6 | — | 10 | — | — |

9·96 volts is in better agreement with the equation for the 5th group atoms than the former. Again, it appears that the value of 8·0 volts assigned to bismuth in the 'International Critical Tables' is a closer approximation to the ionization potential than the value of 7·25 volts ascribed to the element in the "Tabellen." In the halogen family, the value of 10 volts given in the 'International Critical Tables' as the constant for iodine is in better agreement with the

line passing through chlorine and bromine than the estimate of 8 volts found in the other compendium.

The equations permit an estimation of the ionization potentials of several heavy elements. For the neutral atoms these estimates are: 11·6 volts for radon, 3·81 volts for eka-cæsium, 8·0 volts for tellurium, 7·6 volts for polonium, and 10 volts for eka-iodine. The ionization potential of the singly ionized radium atom is 10·23 volts, the value calculated from Equation II. (Table III.) is only 9·53 volts; if this behaviour is typical of all the atoms occurring in the 7th period, then the estimates for radon and eka-cæsium are probably too low. K. T. Bainbridge [5] has estimated the ionization potential of eka-cæsium from the alkali and alkaline earth isoelectronic levels, and ascribes to the constant a value of $4{\cdot}05 \pm 0{\cdot}05$ volts. The following equations express the ionization potentials of the heavy alkali and alkaline earth metals somewhat better than the equations given in Tables II. and III.

Alkalies ..... $n^2V = 9{\cdot}836\ Z^{2/3} - 2{\cdot}58.$

Alkaline Earths... $n^2V = 12{\cdot}60\ Z^{2/3} + 2{\cdot}38.$

By extrapolation, the second equation leads to a value of 5·13 volts for the neutral radium atom, which is only a small improvement on the value calculated in Table III., since the observed ionization potential is 5·4 volts. Extrapolation of the equation for the alkalies gives 3·89 volts as the ionization potential of eka-cæsium. This value just exceeds that of cæsium, but it is well to note that a purely empirical formula, based on the normal trend of the ionization potentials in the alkali group, would yield a value much lower than that of cæsium.

In the case of the singly ionized atoms the variation of $n^2V$ with $Z^{2/3}$ is not as uniform as for the neutral atoms. Only in the cases of the noble gases and the alkaline earths are sufficient data available for the evaluation of the coefficients on averaged points. The slopes of the lines representing these families are 0·7671 and 0·6188 respectively; these figures can be interpreted as approaching the fraction 2/3 only in a very approximate way. In the alkali group the ionization potentials of only lithium, sodium, and potassium are definitely known. An estimate of 16 volts is given in the "Tabellen" for the second ionization potential of rubidium. This value deviates considerably from the family line; the number found by extrapolation is 29 volts, which probably represents the true value, as the same equation predicts the ionization potential of lithium with only a small

error. A recent analysis of the spectra of rubidium [6] and cæsium [7] by Laporte, Miller, and Sawyer reveals that the ionization potential of the singly ionized atoms are 27·3 and 23·4 volts respectively.

*Summary.*

From considerations of the interatomic distance between atoms in crystals, Hume-Rothery has deduced that $n^2V$ is approximately a linear function of $Z^{2/3}$. The present analysis shows that for the neutral atoms the theorem is essentially correct, but that a better correlation of the properties of the members of a specific group is given by the relationship: $\log n^2V = a \log Z + b$, where the coefficient $a$ approaches the fraction 2/3 in the several periodic groups. For singly ionized atoms the latter relationship is also valid, but the values of $a$ exhibit more marked deviations from the theoretical limit than in the case of the neutral atoms.

*References.*

(1) W. Hume-Rothery, Phil. Mag. [7] x. p. 217 (1930).
(2) *Ibid.* [7] xi. p. 670 (1931); [7] xiii. p. 196 (1932).
(3) 'International Critical Tables,' v. p. 392 (1929); vi. p. 70 (1929), 1st ed., N.Y.
(4) Landolt-Börnstein, 'Physikalisch-Chemische Tabellen,' 5th ed., ii. table 152 f, Berlin (1931).
(5) K. T. Bainbridge, Phys. Rev. xxxiv. p. 752 (1929).
(6) O. Laporte, G. R. Miller, and R. A. Sawyer, Phys. Rev. xxxviii. p. 843 (1931).
(7) *Ibid.* xxxix. p. 458 (1932).

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CIX. *The Photoelectric Properties of Films of Beryllium, Aluminium, Magnesium, and Thallium.* *By* H. de Laszlo, *Sir William Ramsay Laboratories of Physical and Inorganic Chemistry, University College, London, and Massachusetts Institute of Technology, Cambridge, Mass.**

Summary.

Opaque films formed by the evaporation of the above metals in high vacuum on to a previously degassed surface were examined with respect to their photoelectric properties in monochromatic light. Measurements were made between λ 5000 and 2400 Å., and the photoelectric response in coulombs

* Communicated by Prof. F. G. Donnan, C.B.E., M.A., F.R.S.

per erg as well as in electrons per quantum was plotted against the wave-length. Magnesium was found to be a suitable surface to use in the construction of photoelectric cells for measurement in the ultra-violet.

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## Object of Investigation.

THE work here described was part of a programme for measuring accurately the photoelectric response (in absolute units) versus wave-length (both visible and ultra-violet) of a number of commercial photocells. The results obtained with these cells, although of interest industrially, were not suitable for scientific publication, owing to the fact that no two cells were reproducible. This was due to the presence of gas, and hence the sensitivity of the metal surface was variable.

Since much time had been expended in devising suitably accurate methods of measurement, we decided to determine the photoelectric properties of a number of elements, chiefly with the object of using such cells in lieu of thermopiles for energy measurements in the ultra-violet in conjunction with quantitative photochemical work, and for use in automatically measuring the variation in concentration of substances in the state of vapour or solution which absorb strongly in the ultra-violet and not in the visible.

Aluminium and magnesium have already been examined [1], but the results showed that there was a constant change in the photoelectric sensitivity, owing to gas being present. Nor could this have been avoided with the technique then available. We have sought to eliminate the factors of error present in the above investigation, and have obtained reproducible results for Be, Al, Mg, and Th.

## Experimental.

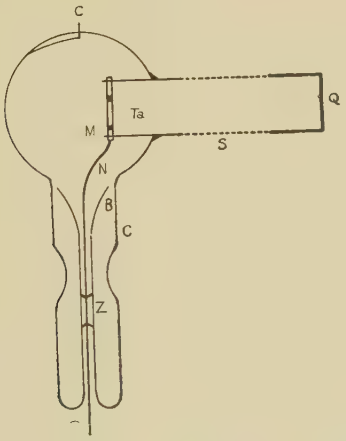
### 1. *Preparation of the Metal Films.*

In determining the photoelectric efficiency of a metal, it is of primary importance that all the incident light should be absorbed by the metal itself. This condition is best realized if the metal to be examined be deposited on the inner surface of a sphere, leaving only a small opening in the sphere for the radiation to enter by. Since the amount of energy available in the form of monochromatic radiation is usually very small the photoelelectric currents are correspondingly small. For this reason it is essential that there should be a

very high leakage resistance between anode and cathode. In order to measure the true photoelectric effect the whole operation of forming the film is conducted in the highest available vacuum so as to avoid occluded films of gas. After many attempts, a cell of the type shown (fig. 1) was developed for these measurements.

The metals examined were the purest type available. Small pieces of metal (M) were spot-welded on to a 2 mm. wide ring of degassed tantalum (Ta), which in its turn was supported by a nickel wire (N) and sealed with tungsten through the pyrex stem (Z). The Ta had been previously degassed by heating for a few minutes at 1000° C. and was

Fig. 1.




then allowed to cool to room temperature in a high vacuum. This procedure removed any occluded gases. The metals to be examined had previously been distilled in high vacuum. The bulb of the cell was made of pyrex, and a quartz window (Q) was sealed into the bulb with the aid of a graded seal (S). On heating Ta with a high-frequency current applied externally the metals present on the ring volatilized and deposited on to the walls of the bulb. Owing to a continuation of the window tube through the ring, none of the metal was able to reach the quartz window. A short piece of platinum strip attached to a piece of tungsten wire (C) was melted on to the inner surface of the bulb in order to provide electrical contact with the film. The upper part of the anode stem was slightly belled out (B), while the outer

part of the stem was contracted below (B). In this way no metal could be deposited below the seal (Z). The leakage resistance of the cells used was of the order of $10^{-13}$ amps. Before heating Ta the entire cell was baked out at 400° C. for several hours while attached to a 3-stage quartz Volmer mercury pump. A liquid air-trap was provided between the seal and the pump. In this way we hoped to remove occluded gases that might be present on the interior of the bulb. When the experiment was ended the stem was cracked off at (C), a new charged Ta ring was welded on to the nickel, the interior cleaned with hot nitric and chromic acids, and the stem re-sealed on to the bulb. Sufficient metal was evaporated to render the film opaque.

### 2. *Method of measuring small Photoelectric Currents.*

The Wheatstone Bridge valve method of measuring small currents developed by Wynne-Williams[2] was tried, but found to be insufficiently sensitive owing to the small currents available. In the end a modified Rosenberg[3] method was evolved. This method consists of measuring the drop of potential through a high resistance, R, in which the photocell current flows. As long as this drop does not exceed 1 or 2 volts, the impedance of the resistance is negligible compared to that of the photocell. Resistances of the order of $10^8$ to $10^{11}$ ohms were necessary, and after trying a number of types, that described by Gyemant[4] was found to be the most satisfactory. These resistances consisted of glass tubes 4 ins. in length and $\frac{1}{2}$ in. in diameter, with a platinum wire sealed into either end, filled with a mixture of pure benzol containing 0·7 per cent. of picric acid by weight. Pure ethyl alcohol is added in sufficient quantity to give the desired resistance.

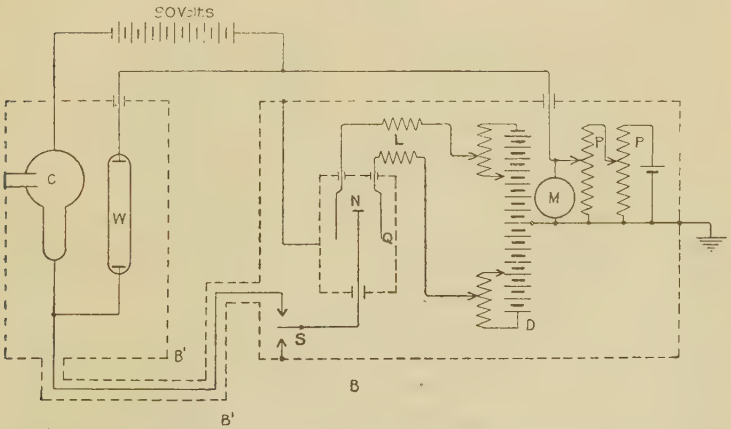
These resistances were calibrated against a 200,000 ohm standard with the aid of a sensitive galvanometer, and had the following values: $3{\cdot}56 \times 10^7\,\Omega$, $1{\cdot}46 \times 10^9\,\Omega$, $1{\cdot}50 \times 10^{11}\,\Omega$. They showed a very low temperature coefficient, the variation being less than 0·2 of 1 per cent. over the range 15–20° C.

The circuit used is shown in fig. 2 and is self-evident. Any current flowing in the circuit causes a drop of potential in the resister W. This charges and deflects the needle, N, of a Lindemann electrometer [5], which is returned to zero with the aid of a potentiometer, P. The balancing voltage, V, across P, is read by means of a precision millivoltmeter, M. Knowing V and resistance W, the current can be calculated.

The entire circuit, including electrometer and microscope for reading the same, was most carefully shielded by placing in an iron box, B, from which only the eyepiece of the latter protruded.

The photocell and high resistance were mounted in a separate metal box, B, the lead from the cell anode to the electrometer being carefully insulated with a quartz tube and shielded with metal braid. The box and contents were mounted on a rider which could slide along an optical bench placed at right angles to the exit beam of the monochromator. Currents greater than $10^{-13}$ amperes could be measured with precision, and $10^{-14}$ could be detected.

Fig. 2.




C, photoelectric cell; W, high resistance; S, earthing switch; Q, electrometer quadrants charged to about 30 volts; N, electrometer needle; L, Grid leaks 1–2 megohms; P, potentiometers about 2000 ohms; D, 90-volt battery.

### 3. *Source of Light.*

In order to reach a high degree of accuracy in our measurements it was necessary to have a strong source of monochromatic ultra-violet radiation. If this were weak, although a reading might be obtained with the photocell, yet the radiation energy would not be sufficient to give good readings with a thermopile.

The constricted mercury arc, as developed by Forbes[6], was found to be suitable when run at 1 ampere and 100 volts. The discharge took place in a quartz capillary 2 mm. I.D. and about 25 mm. long. This was placed very close to the

slit of the monochromator. To keep the arc steady a large inductance (several hundred Henries) was placed in series with it, and a controlled blast of air was directed on to the capillary. After burning for about 10 hours the quartz devitrifies to some extent and a yellow coating forms in the

TABLE.

| | Beryllium. | | Aluminium. | | Magnesium. | | Thallium. | |
|---|---|---|---|---|---|---|---|---|
| A. | coul. ergs $\times 10^{-13}$. | electron quantum. | coul. ergs $\times 10^{-12}$. | electron quantum. | coul. ergs $\times 10^{-12}$. | electron quantum. | coul. ergs $\times 10^{-13}$. | electron quantum. |
| 5791 | — | — | — | — | — | — | — | — |
| 5461 | — | — | — | — | — | — | — | — |
| 4347 | ·8 | ·0000021 | — | — | 21·6 | ·00061 | — | — |
| 4039 | 3·9 | ·000012 | — | — | — | — | — | — |
| 3655 | 24·3 | ·000084 | ·8 | ·000027 | 57·7 | ·0020 | — | — |
| 3341 | 91·5 | ·00034 | 2·4 | ·000088 | — | — | — | — |
| 3130 | 148 | ·00056 | 3·8 | ·00015 | 252·0 | ·010 | 2·4 | ·0000096 |
| 3022 | 182 | ·00075 | 4·9 | ·00020 | — | — | 6·3 | ·000026 |
| 2967 | 201 | ·00083 | 5·6 | ·00023 | 427·0 | ·018 | 10·1 | ·000042 |
| 2893 | — | — | 6·5 | ·00028 | 540·0 | ·023 | — | — |
| 2806 | 290 | ·0013 | 7·8 | ·00034 | 738·0 | ·033 | 21·7 | ·000093 |
| 2699 | 359 | ·0016 | 12·0 | ·00053 | — | — | 30·8 | ·00014 |
| 2653 | — | — | | | 1200·0 | ·056 | 38·9 | ·00018 |
| 2536 | 331 | ·0016 | 12·7 | ·00062 | 1330·0 | ·065 | 52·8 | ·00026 |
| 2483 | 319 | ·0016 | 11·9 | ·00060 | 1260·0 | ·063 | 52·8 | ·00026 |
| 2399 | — | — | 0 | 0 | 792·0 | ·041 | 44·0 | ·00023 |

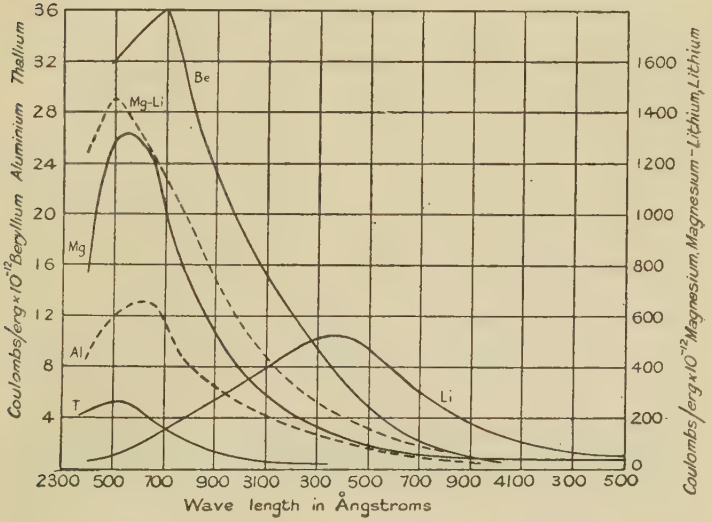
interior; the latter could be removed with hydrofluoric acid, but owing to the former, the capillary must be renewed from time to time. The monochromator was of the constant deviation type in which a triple quartz [7] prism was caused to rotate in a novel way that has already been described elsewhere [8]. The lenses were quartz-fluorite achromatic

triplets, 30 mm.[9], and were free from any visible fluorescence.

### 4. *Radiation Energy Measurement.*

The radiant energy incident on the photocell was always measured in absolute units before and after each photoelectric observation. This was done by pushing a thermopile mounted on a rider along the same optical bench on which the photocell was mounted. Thus either the cell or the pile would be placed quickly and accurately in the path of the light. A photographic shutter was mounted in front of the exit lens of the monochromator. The thermopile consisted

Fig. 3.




of 17 elements of copper and bismuth and had a silver target which measured $20 \times 2\frac{1}{2}$ mm. It was calibrated against a standard lamp (Bureau of Standards, Washington, D.C.), and its sensitivity was found to be

$$\frac{\text{energy}}{\text{mm. deflexion}} = \frac{0{\cdot}39 \times 10^{-6} \text{ watts}}{\text{mm.}}.$$

## Measurements.

Our results are tabulated in the Table. We have included values for what might be called the true photoelectric efficiency[10] of the surface in question at each wave-length, *i. e.*, electrons emitted per quantum of incident energy. The

figures show how inefficient the whole process is. On fig. 3 we give curves of photoelectric efficiency versus wave-length. The curve for a commercial quartz-lithium cell is shown as well as for that obtained for an 80–20 per cent. Mg-Li alloy. The two latter are given for comparative purposes, as one set of measurements only were made for their determination. The results are accurate to at least $\pm 5$ per cent.

## Conclusion.

Be and Al would not be of great use in cell construction, owing to their low efficiency and to the fact that their spectral sensitivity is covered by that of the more efficient magnesium.

Th, although very insensitive, might be of use for certain purposes. It is responsive to the ultra-violet only, and cuts off sharply at 3400 Å.

Magnesium appears to be a very suitable material to use in the construction of ultra-violet photocells. Its efficiency between 2800 Å. and 2400 Å. is exceptionally high, and it would be of practical use as far down as 4000 Å. Such cells would be simple to construct owing to the ease with which Mg can be distilled in high vacuum.

The curve of the Mg-Li alloys shows that there is every possibility of constructing cells having any desired spectral sensitivity distribution. Thus alloys of Mg with any varying proportions of one or more of the alkali metals might be tried.

In conclusion, I should like to thank Mr. Poitras and Mr. St. Louis for their able assistance in the experimental part of this work.

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(1) Pohl and Pringsheim, *Verh. d. D. Phys. Ges.* x. p. 548 (1912).
(2) Wynne-Williams, Phil. Mag. vi. p. 324 (1928).
(3) Rosenberg, *Z. f. Phys.* vii. p. 18 (1921).
(4) Gyemant, *Wiss. V. d. Siemens Konzern*, p. 58 (1928).
(5) Lindemann and Keeley, Phil. Mag. xlvii. p. 577 (1924).
(6) Forbes and Harrison, J. Opt. Soc. Am. xi. p. 99 (1925).
(7) Bernhard Halle, *Nachf. Berlin-Steglitz*,
(8) de Laszlo, J. Sci. Ins. vii. p. 292 (1930).
(9) 'Optique Scientifique,' 65 Rue Halle, Paris.
(10) de Laszlo, "Photoelectric Cells and their Application," Phys. and Opt. Soc. Discussion, p. 221 (1930).

CX. *Discharges maintained by Electrical Oscillations in Solenoids.* *By* G. D. Yarnold, *B.A., Merton College, Oxford* *.

(1) IT is well known that the electric force inside a solenoid, which forms part of an oscillating circuit, consists of two distinct parts: a force due to the difference in potential E between the ends of the solenoid, and a force proportional to the rate of change of the magnetic field. The force due to the potential difference E may be referred to as the "electrostatic force," and the force due to the change of the magnetic field as the "electromagnetic force."

The relative magnitudes of these two forces will obviously depend upon the dimensions of the solenoid. A simple calculation shows, however, that in solenoids of the dimensions of those generally used for exciting high-frequency discharges in gases, the electrostatic force is much greater than the electromagnetic force. Thus, in a solenoid of radius 5 cm., with turns 1 cm. apart, the electrostatic force is about thirty times as large as the electromagnetic force †.

High-frequency currents are obtained in rarefied gases when the tube containing the gas is placed inside the solenoid. The currents may be excited either by damped oscillations obtained with condenser discharges, or by continuous oscillations maintained by valves.

An account of the earlier experiments on electrodeless discharges, which were made with damped oscillations, is given by Mierdel in the *Physikalische Zeitschrift* ‡.

Many early investigators supposed that the electrodeless discharges excited by the high-frequency currents in a solenoid are due to the electromagnetic force. This view was held by Hittorf, who first obtained electrodeless discharges in 1884.

Lecher, however, showed that when the electrostatic force is reduced by means of a screen consisting of strips of tinfoil, inserted between the solenoid and the bulb containing the gas, it is impossible to obtain a discharge. He was thus led to the conclusion that the electrodeless discharges excited by the currents in the solenoid are due to the electrostatic force.

In these early experiments with damped oscillations, it was difficult to make any measurements either of the potential

* Communicated by Prof. J. S. Townsend, F.R.S.

† J. S. Townsend and R. H. Donaldson, Phil Mag. v. p. 178 (1928).

‡ G. Mierdel, *Phys. Zeit.* xxv. p. 240 (May 1924).

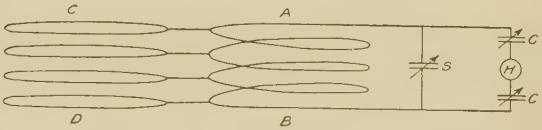
difference E between the ends of the solenoid, or of the current flowing in the solenoid.

More recently, however, Townsend and Donaldson, using continuous oscillations, have made accurate measurements of the potential differences necessary to excite electrodeless discharges under various conditions, and have come to the same conclusions as Lecher.

In the last two or three years, other physicists * have also made experiments with continuous oscillations, and have sought to explain the results of their experiments on the hypothesis that the discharge is of electromagnetic origin.

As their arguments do not appear to be conclusive, it was considered desirable to make further experiments in order to find the relative importance of the electrostatic and electromagnetic forces. For this purpose, I have made two different investigations with continuous oscillations, both of which give very definite results.

Fig. 1.




(2) In the first of these investigations, I have observed a discharge in an electrostatic field and compared the results with those obtained in the field inside a solenoid.

The apparatus used is shown in fig. 1. The solenoid AB was 10 cm. long and had thirteen turns of wire of 19 cm. diameter. For simplicity, four turns are shown in the figure. It was loosely coupled with a continnous wave generator and was tuned by the condenser S. The strength of the oscillations induced in the solenoid was varied by adjusting this condenser. The potential difference E between the ends of the solenoid was measured by means of the thermo-milliammeter H in series with two condensers of equal capacity C, in the same way as in the experiments of Townsend and Donaldson.

The experiments recorded here were all made with oscillations of wave-length 97 metres; and the potential

* K. A. MacKinnon, Phil. Mag. xiii. p. 605 (Nov. 1929); C. T. Knipps, Phys. Rev. xxxvii. p. 756 (March 1931); H. Smith, W. A. Lynch, N. Hilbery, Phys. Rev. xxxvii. p. 1091 (May 1931); F. Esclangon, *Comptes Rendus*, cxcii. p. 1213 (May 1931).

differences given are the root mean square values expressed in volts.

The method of obtaining the same electrostatic field as that inside the solenoid, but without the accompanying electromagnetic field, was similar to that used by Esclangon. Alternate turns of the solenoid were connected to seven equally-spaced coaxial wire rings of the same diameter as the turns of the solenoid, and forming a cage CD of the same length as the solenoid. Each ring was at the same potential as the point on the solenoid to which it was connected, and therefore the electrostatic force inside the cage was approximately the same as that in the solenoid.

The discharges were produced in a bulb of 12 cm. diameter, containing neon at a pressure of about 1 mm.

(3) It was found that there is no discharge in the gas until the potential difference between the ends of the solenoid reaches a certain value $E_1$, at which the discharge starts

TABLE I.

| | Solenoid. | Cage. |
|---|---|---|
| $E_1$ | 245 | 260 |
| $E_2$ | 118 | 125 |

suddenly. If the potential is increased, the current in the gas increases, and if the potential is decreased, the current decreases. There is a certain minimum potential $E_2$, however, below which it is impossible to maintain even a small current in the gas,

When the bulb is held inside the cage the same phenomena are observed. The values found for $E_1$ and $E_2$ when the bulb was in the solenoid and when it was in the cage are shown in Table I.

The fact that the potentials $E_1$ and $E_2$ obtained with the bulb in the solenoid are nearly the same as those obtained with the bulb in the cage shows that the electromagnetic force is unimportant in starting or maintaining the discharge.

(4) In the second investigation the electrostatic force inside the solenoid was reduced by means of screens placed inside the solenoid, as in Lecher's experiments.

Four screens were made consisting of strips of tinfoil 5 mm. wide pasted on the inside of paper cylinders, in a

direction parallel to the axis. The lengths and distances apart of the strips in the four screens were as follows :—

| | | | | | | |
|---|---|---|---|---|---|---|
| I. | Strips | 30 cm. | long and | 5 mm. | apart. | |
| II. | „ | 30 cm. | „ | „ 1 mm. | | „ |
| III. | „ | 15 cm. | „ | „ 5 mm. | | „ |
| IV. | „ | 15 cm. | „ | „ 1 mm. | | „ |

When a screen was inserted in a solenoid it was found that there is no discharge in a bulb placed inside the screen when the potential difference between the ends of the solenoid is equal to the potential $E_1$ given in Table I. The discharge does not start until the potential is raised to a value $F_1$ much larger than $E_1$. In the same way, the minimum potential $F_2$ which will maintain a discharge in the gas is much larger than $E_2$. The values found for $F_1$ and $F_2$, when screen III. is used, are shown in Table II., with the values of $E_1$ and $E_2$ for comparison.

Table II.

| | | | |
|---|---|---|---|
| $F_1$ | 2500 | $E_1$ | 245 |
| $F_2$ | 860 | $E_2$ | 118 |

It is seen from the table that when this screen is used the potential $F_1$, which is necessary to start a discharge, is over ten times as great as the potential $E_1$, which is necessary when no screen is used ; while to maintain a discharge the potential $F_2$ must be over seven times as great as the potential $E_2$.

With any of the screens except No. III. a potential difference of 2700 volts between the ends of the solenoid is insufficient to start a discharge in the bulb.

So long as the potential difference between the ends of the solenoid is as high as 2000 volts, however, it is possible to start the discharge by holding the bulb outside the screen, and then to transfer the bulb to its central position inside the screen, without stopping the discharge. In this way the minimum potential $F_2$, which will maintain a discharge when the bulb is inside the screen, can be found for each of the other three screens. The values obtained for $F_2$ with the different screens are shown in Table III.

(5) These results, and those given in Table II., show that when the electrostatic force inside the solenoid is reduced by means of one of the screens, then the potential difference

which must be applied between the ends of the solenoid in order to start a discharge, or to maintain a small current, is very much larger (in some cases more than twelve times larger) than the potential necessary when no screen is used.

In the first investigation it is shown that the potential which is required to start a discharge when the bulb is inside the cage, where there is no electromagnetic force, is practically the same as that required when the bulb is inside the solenoid.

Both investigations show that the discharge obtained when the bulb is inside the solenoid is due almost entirely to the electrostatic force.

The question arises whether the discharges obtained with the screens are maintained by the electromagnetic force, or by an electrostatic force due to incomplete screening.

Referring again to the results shown in Table III., it is seen that the potential $F_2$, which is required to maintain a small current in the gas, is not constant, but depends on the

TABLE III.

| Screen. | I. | II. | III. | IV. |
|---|---|---|---|---|
| $F_2$ | 1160 | 1250 | 860 | 960 |

screen used. The value found for $F_2$ is lower with screen III. than with any other screen ; and, similarly, the value of $F_2$ is higher with screen II. than with any other screen. From its construction we should expect screen II. to be the most effective, and screen III. the least effective.

The fact that the potential $F_2$, necessary to maintain a small discharge, varies from screen to screen shows that the discharge obtained when the bulb is inside a screen is due, at any rate partly, to an electrostatic force arising from incomplete screening.

(6) That the discharge is also due partly to the electromagnetic force in this case is shown by measuring the potential $G_2$, which will maintain a discharge when the bulb is inside the cage, and a screen is inserted in the cage. Any discharge obtained under these conditions will be due entirely to the electrostatic force arising from incomplete screening.

In all cases it was found that the potential $G_2$ is greater than $F_2$ for the same screen. With screens I. and II. it is impossible to maintain a discharge, even though the potential

difference between the ends of the cage is as high as 2700 volts, and with no screen is it possible to start a discharge. The values found for $G_2$, with the values of $F_2$ for comparison, are shown in Table IV.

The difference between the potentials $G_2$ shown in Table IV. and the corresponding potentials $F_2$ can only be due to the electromagnetic force, which exists in the solenoid but not in the cage ; while the fact that it is possible to maintain a discharge when a screen is used inside the cage is further evidence that the screening is not perfect, and that the discharges obtained when the screens are used inside the solenoid are due partly to an electrostatic force.

(7) As has been mentioned already, we should expect from the theory of the electric field that the electromagnetic force inside the solenoid used in these experiments would be much less than the electrostatic force. In order to compare these

Table IV.

| Screen. | I. | II. | III. | IV. |
|---|---|---|---|---|
| $G_2$ | Over 2700 | Over 2700 | 1180 | 1500 |
| $F_2$ | 1160 | 1250 | 860 | 960 |

forces experimentally, measurements have been made of the electromotive force $e$ induced in a circular loop of wire held in a central position inside the solenoid, with its plane perpendicular to the axis. The electromagnetic force in the bulb is then found by dividing the electromotive force $e$ by the circumference of the loop.

If a short length of high-resistance wire, to which is soldered a thermo-couple, is included in the loop, it is possible to measure the current flowing in the loop, and so to calculate the electromotive force $e$. For, if L and R are the inductance and resistance respectively of the loop, then the current $i$ is given by

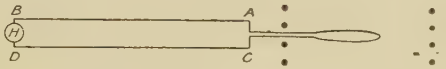
$$i = \frac{e}{\sqrt{L^2p^2 + R^2}},$$

where $i^2$ and $e^2$ are the mean squares of the current and the electromotive force and $p$ is the angular frequency of the oscillations. In these experiments $Lp$ was much larger than R, so that the electromotive force is given by

$$e = i.L.p.$$

It was found that the current flowing in a loop of wire of about 9 cm. diameter is so large as considerably to reduce the potential difference between the ends of the solenoid, when the latter is kept tuned to resonance with the oscillator. Accordingly, the currents in the loop were reduced by adding extra inductance in the form of two parallel straight wires, AB and CD, placed outside the solenoid in such a position that there was no electromotive force induced in them by the currents in the solenoid. This arrangement is shown in fig. 2, where the dots represent a section of the solenoid. The connexions between the two parallel wires and the loop passed between two turns of the solenoid, and a thermo-milliammeter H was included in the circuit between B and D.

Fig. 2.




The values thus found for the electromotive force $e$ induced in a circular loop of 9 cm. diameter are given in volts in Table V., the potential difference between the ends of the solenoid being adjusted to the values of $F_2$ given in Table III.

It is seen that when the potential difference between the ends of the solenoid is just sufficient to maintain a discharge,

Table V.

| Screen. | I. | II. | III. | IV. |
|---|---|---|---|---|
| $e$ | 20 | 22 | 15 | 16 |
| $F_2$ | 1160 | 1250 | 860 | 960 |

the electromagnetic force inside the screen at 4·5 cm. from the axis ($e/9\pi$) is in no case as great as 1 volt per cm. In the case when no screen is used the electromotive force $e$ is 1·6 volts and the electromagnetic force is less than 0·1 volt per cm., whereas the potential difference $E_2$ between the ends of the solenoid is 118 volts. The maximum value of the electrostatic force is therefore about 12 volts per cm. (since the length of the solenoid is 10 cm.). Even supposing that the electrostatic force at 4·5 cm. from the axis is half this, namely, 6 volts per cm., it is seen that the electrostatic force is at least sixty times as large as the electromagnetic force when there is no screen.

(8) From the foregoing evidence we must conclude that the force inside a solenoid which is sufficient to start a discharge, or to maintain a small current in the gas, is mainly electrostatic. This conclusion is borne out by the experiments of Lecher and of Townsend and Donaldson, to which reference has already been made.

The force at any point in the gas when the discharge is taking place, however, is not the same as the force at the same point when the bulb is removed, owing to the modification * of the field by the charges in the gas.

It has been found that in the uniform luminous columns of high-frequency discharges in wide tubes the force in the gas is independent of the current. Similarly, in discharges in bulbs placed inside a solenoid the electrostatic force in the gas does not exceed the force required to maintain a very small current. When the potential E between the ends of the solenoid is increased to values much greater than the potential required to maintain a small current, a large current is obtained in the gas, but there is no corresponding increase in the electrostatic force in the gas. The increase in the current in the solenoid required to increase the potential E is accompanied by a proportional increase in the electromagnetic force. Thus in experiments where large currents are maintained in the gas the electromagnetic force may become comparable with the electrostatic force. This would account for the difference observed by Esclangon between the intensity of the light emitted by discharges in bulbs placed inside a solenoid and the intensity of the light from discharges in bulbs placed in a field where there was no electromagnetic force.

In conclusion, I should like to thank Professor Townsend for his helpful advice and criticism throughout this work.

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CXI. *Similitude and the Heatflow through a Granulated Material.* *By* O. A. Saunders, *M.A., M.Sc.* (*Fuel Research Division, Department of Scientific and Industrial Research*) †.

MR. J. H. AWBERY'S "Note on Heatflow through a Granulated Material" ‡ is an interesting example of the use of the principle of similitude in heat problems. His

* J. S. Townsend, Phil. Mag. xiii. p. 745 (April 1932).

† Communicated by the Author.

‡ Phil. Mag. xii. p. 1152 (December 1931).

conclusions are, in fact, particular cases of the more general result given below, which follows immediately from the principle.

§ 1.

Consider the heat-flow by conduction in any heterogeneous body, with given surface temperatures. The principle of similitude can only be applied to similar systems, *i. e.*, to systems of fixed shape but differing in size. Further, in the present example, the distribution of the impressed surface temperatures must be fixed, while the absolute values are varied ; also the distribution of the thermal conductivity within the body (if not uniform) must be fixed while the absolute values are varied. If these conditions are fulfilled, the principle of similitude shows that the effective thermal conductivity, measured between any corresponding pairs of points, is independent of the size and surface temperatures and is proportional to the conductivity at any specified point.

These results follow directly from the principle, since the size $l$, surface temperature $\theta$, and conductivity $k$, which have dimensions L, $\theta$, and $MLT^{-3}\theta^{-1}$ respectively, cannot be arranged to form a dimensionless group ; the systems are therefore similar whatever the values of $l$, $\theta$, and $k$. The effective conductivity $k'$, measured between any pair of corresponding points, can be arranged with $l$, $\theta$, $k$ in the dimensionless group $k'/k$ only ; hence $k'$ is independent of $l$, $\theta$, and proportional to $k$. It is assumed that the conductivity $k$ is independent of the temperature.

The cases considered by Mr. Awbery of similar solids in contact, with the intervening spaces either conducting or non-conducting, are particular examples of the above result, as also is Prof. Lees's case of a material made in check-board fashion with the black and white squares of different conductivities, the heat-flow being in the plane of the board. It also follows that, in Mr. Awbery's case, provided the ratio of the conductivity of the solids to that of the material filling the intervening spaces is constant, the effective conductivity is proportional to the conductivity of the solids.

The general result is applicable to any shapes or conductivity distributions, and to any distribution of impressed surface temperatures.

§ 2. *Surface Heat Transfer.*

Another case of some interest is that of a system of perfectly conducting bodies, between which heat is transferred either directly from surface to surface or between each surface and an intervening medium, the heat transfer per

unit surface area being proportional to the temperature difference and independent of the distance apart of the surfaces. In this case the principle of similitude shows that, for similar systems, the effective conductivity is independent of $\theta$ and proportional to $\alpha l$, where $\alpha$ is the surface heat transfer coefficient This result is evident from the fact that $l$, $\theta$, $\alpha$ cannot be arranged in a dimensionless group, while the dependent variable $k'$ can only be grouped with them in the arrangement $k'/\alpha l$.

### § 3. *Application to a Granulated Material.*

§ (1) shows that if the heat transfer across the air spaces in a granulated materal were by conduction alone, the effective conductivity would be independent of the linear size of the grains, for geometrically similar grain structures. The effective conductivity would not be proportional to the conductivity of the grain material unless the ratio of the conductivity of the air to that of the grains were constant. Mr. Awbery has since pointed out, in correspondence with me on the subject, that since the conductivity of air is very small in comparison with that of most solids, it can often effectively be taken as zero. In this case the effective conductivity would be proportional to that of the solid material. These conditions are approached for very small air spaces, in which convection is restricted, while radiation is of decreasing importance compared with conduction, especially at low temperatures.

On the other hand, if the heat transfer across the spaces were independent of their distance apart, and if the grains were perfect conductors, § (2) shows that the effective conductivity would be directly proportional to the grain size. The larger the air spaces, especially at higher temperatures, and the higher the conductivity of the grains, the more nearly are these conditions approached.

Practical cases must fall somewhere between these two extremes.

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## CXII. *Note on the Electrocapillary Effect of Capillary-active Organic Molecules.* *By* A. W. Davis, *B.Sc., Mardon Research Scholar, Washington Singer Laboratories, University College, Exeter* *.

THE electrocapillary curve of a solution of a capillary-active electrolyte is approximately parabolic. If to such a solution be added a capillary-active non-electrolyte,

* Communicated by Prof. F. H. Newman, D.Sc.

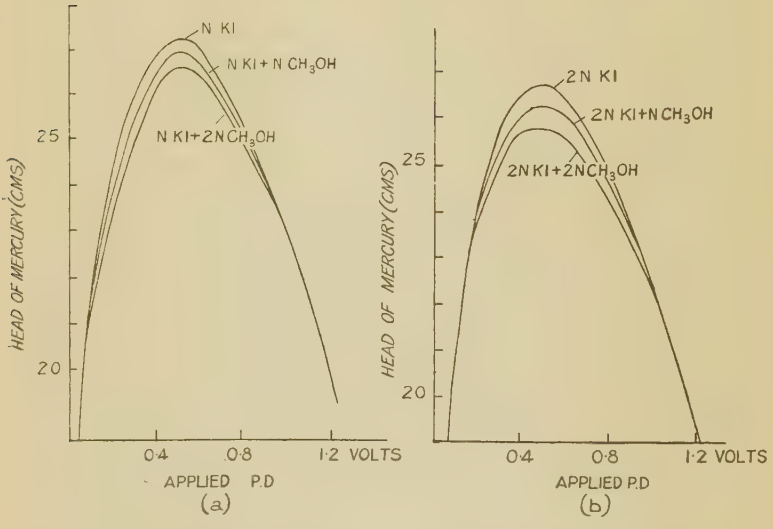
*e.g.*, one of the alcohols, and the electrocapillary curve of the mixed solution be obtained, the second (modified) curve is found to be depressed in the region of the maximum, but on either side of the maximum ordinate the curves merge into one another and finally coincide—the modified curve appears to be a truncated replica of the primary curve. Gouy * discovered this effect of non-electrolytes, and it has since been studied by Frumkin †, Butler ‡, Ockrent §, and Wightman ‖. The depression of the interfacial tension has been shown to be due to adsorption and orientation of the inorganic molecules in the mercury-solution interface ¶. If, however, the electrolyte in the primary solution yields an active ion on dissociation, then one branch of the electrocapillary curve is shifted over its entire length, and the maximum with it, *e.g.*, with KI the active I-ion depresses both the left-hand branch of the curve and the maximum, the right-hand branch coinciding with that of such an inactive solution as KCl.

In the present work the effect of this active ion upon the adsorption of active organic molecules in a mixed solution was investigated. Normal and twice-normal solutions of KI were prepared, and their electrocapillary curves obtained. Then to each solution was added sufficient $CH_3OH$ to give a normal and twice-normal solution of the latter substance, and the electrocapillary curves of the four mixed solutions so formed were obtained. Measurements were made with the aid of an ordinary Lippmann electrometer. The fine capillary tubes were drawn out from pieces of ·5 mm. bore thick-walled capillary tubing, and were attached to the vertical tube of the electrometer by means of a short piece of pressure tubing. Readings were taken at room-temperature which varied from 15° to 17° C. The methyl alcohol used was pure and free from acetone, and the solutions were saturated with mercurous iodide, the large electrode being covered with a thin layer of HgI to eliminate polarization at its surface. The potentials were applied by means of a resistance-box potentiometer and an accumulator, the accumulator being frequently standardized by balancing against a Weston standard cell.

The resulting curves for the mixed solutions were superimposed on those of the corresponding KI solutions (see figure),

* *Ann. Chim. Phys.* viii. p. 291 (1906).
† *Zeits. Physik.* xxxv. p. 291 (1926).
‡ Proc. Roy. Soc. cxxii. p. 399 (1929).
§ Journ. Phys. Chem. xxxiv. p. 2297 (1930).
‖ *Ibid.* xxxv. p. 3293 (1931).
¶ Frumkin, 'Colloid Symposium Annual,' vii. (1929).

and the former were found to be of the "truncated" type of Gouy, that is, of a similar type to those obtained for solutions of capillary-inactive electrolytes when modified by the presence of surface-active organic molecules. In the region of the maximum the molecules of alcohol cause a depression of the interfacial tension both for the KI and KCl solutions, the depression being due to the adsorption and orientation of the organic molecules in the interfacial phase separating the solution and the mercury. With applied potential differences higher than that corresponding to the maximum, the interfacial tension depression decreases as the




negative potential across the interface increases until, with a certain applied potential difference depending on the relative concentration of the two substances in solution, the primary and modified curves merge into one another, the falling branches of the curves then coinciding with that due to the potassium ion alone. With applied potential differences less than that corresponding to the maximum, the effect is slightly different, since on this positive side of the electrocapillary curve both the active ions and the active molecules have a tendency to be adsorbed at the interface, but again the curves finally coincide. Thus, as the interfacial potential difference increases positively, *i.e.*, as we move away to the left of the maximum ordinate, the effect of the

active organic molecules decreases, and that of the ions increases relatively, until the $CH_3OH$ molecules adsorbed at the interfacial phase, when the interfacial tension is near its maximum value, are replaced entirely by the active I-ions, and the curves follow the same path as for the pure KI solution. The alcohol thus causes a depression of the interfacial tension at the maximum of the electrocapillary curve, but the total depression with the mixed solutions is much less than the sum of the depressions caused by the I-ions and the $CH_3OH$ molecules acting separately, *i.e.*, the depression produced by the alcohol in the KI solution is much less than that which it causes in a corresponding KCl solution; in the neighbourhood of the maximum, the iodide ions affect the values of the interfacial tension, so that the modified curves begin to follow more and more closely those of the corresponding KI solution until they become identical.

Thus the modifying effects of the active organic molecules are similar whether the primary solute is an active electrolyte or an inactive one, except that the active ions determine the form of that branch of the curve corresponding to their normal adsorption. This is in accordance with the results of Butler and Wightman * on the effect of capillary-active electrolytes on the adsorption curves of solutions containing surface-active substances, and with those of Butler and Ockrent † on the adsorption curves of mixtures of surface-active salts and non-ionized molecules. The latter workers found that the effect of the mixture is greater than that of either component alone, but much less than the sum of their separate effects.

An attempt was made to repeat Smith's work ‡ on the electrocapillary curves of aqueous solutions of capillary-active and inactive solutes, using alcohol as the solvent instead of water. Gouy § found that when he used a Lippman electrometer in such alcoholic solutions the capillary jet soon developed a "sticking" effect (which he termed "démouillage"); Newberry ‖ attempted to measure single potential differences in alcoholic solutions using such an electrometer, and he also found great difficulty in obtaining a fine capillary jet that would function, though he records ultimate success.

In the present work the tubes were drawn out as described

* *Loc. cit.*
† *Loc. cit.*
‡ Phil. Trans. cxciii. p. 47 (1900).
§ *Ann. Chim. Phys.* (7) xxix. p. 145 (1903).
‖ Journ. Chem. Soc. p. 2553 (1914).

above, and great care was taken to keep them clean; but, using methyl alcohol and ethyl alcohol solutions, not one was found in which the small meniscus, between the mercury and the alcoholic solutions, retained its mobility for more than a few minutes. In aqueous solutions, the development of such a sticking effect can be checked by allowing the fine tubes to stand in distilled water; but although jets were left in pure alcohol for several days, the mercury meniscus still lost its mobility in a few minutes, so that only unreliable results were obtained. No successful attempt has been made to account for this effect; Gouy merely stated that alcohol developed a contact angle with glass, and abandoned further work with pure organic solvents.

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CXIII. *The Dielectric Constant of Nitrogen up to* 150 *Atmospheres at* 25°, 75°, *and* 125° C. *By* A. *and* C. Michels, *Natuurkundig Laboratorium der Gemeente-Universiteit, Afd. Thermodynamica, Amsterdam**. (29th Communication of the Van der Waals Fund.)

THE dielectric constant of nitrogen at atmospheric pressure has been measured at several temperatures [1] and at pressures up to 226 atm. at room temperature [2], but no measurements have been made over a wide temperature range at higher pressures. The present paper gives the first results which have been obtained with an apparatus designed to measure the combined influence of pressure and temperature on the dielectric constant.

The dielectric constant was obtained from the change in capacity of a condenser containing the gas under investigation. This change in capacity was measured by the standard heterodyne beat method. One high-frequency circuit included the pressure condenser and a calibrated variable condenser, and the other was stabilised by means of piezo-electric quartz. The beat note from these circuits, after detection and amplification, was observed with a loud-speaker together with the note from a tuning-fork. The tuning-fork was kept in vibration in the ordinary way by means of a triode valve. During the measurements the variable condenser was set so that the audible beat between the heterodyne note and the tuning-fork disappeared. It was arranged that the capacity of the pressure and variable condensers could be compared with a normal set.

* Communicated by Prof. F. G. Donnan, C.B.E., M.A., F.R.S.

The pressure condenser consisted of two "precision" tubes insulated from each other with quartz and placed inside a steel vessel. The outer tube was connected to the latter and a lead from the inner tube brought through the steel wall with quartz insulation. As a result of this arrangement the condenser was subjected to hydrostatic pressure, and the change of the zero-capacity due to the deformation of the materials could be calculated directly from the linear compressibility. The correction due to temperature changes could also be calculated from the expansion coefficient of the material. The steel pressure vessel was placed in a thermostat kept constant within 0·01°.

The zero-capacity of the condenser was calculated from the change in the capacity when the condenser was evacuated and filled with benzene. In order to avoid dismantling the apparatus the benzene was distilled into the evacuated condenser. The value for the dielectric constant of benzene (2·259 at 25°) was taken from the work of R. T. Lattey and O. Gatty [3]. The zero-capacity of the condenser was found to be 1443 cm.

Measurements were made with nitrogen guaranteed to contain less than 0·01 per cent. oxygen but not entirely free from inert gases. The pressure was measured with a spring manometer calibrated against a pressure balance. The experiments were carried out using a frequency of 508 kilocycles ($\lambda = 591$ metres). The results are given in Table I. and the accompanying figure.

Table I.

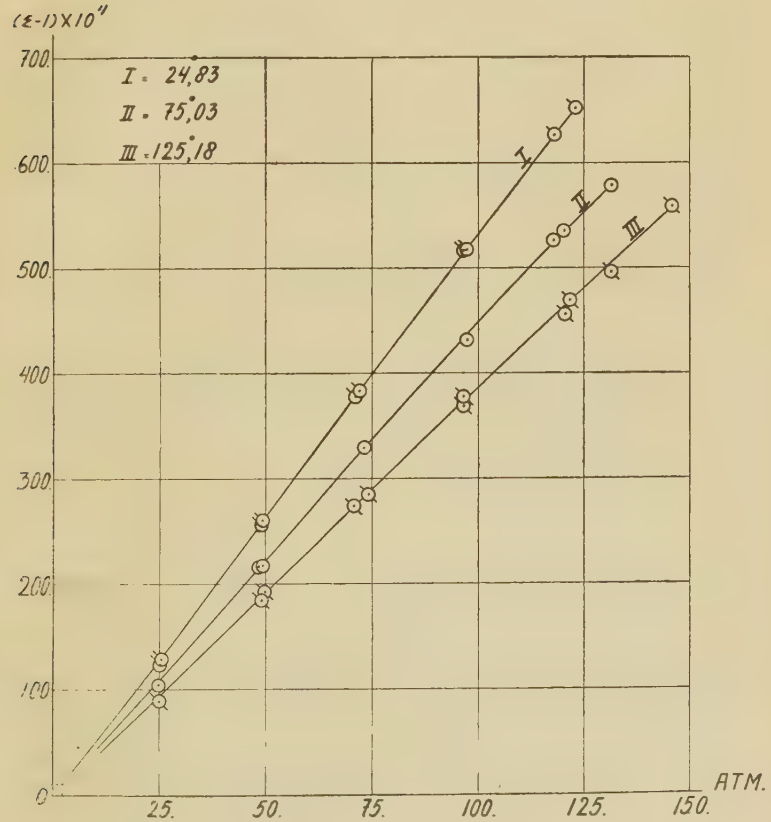
| $P_{Atm.}$ | $d_{Amagat.}$ | $\epsilon$. | $\frac{\epsilon-1}{\epsilon+2} \cdot \frac{1}{d} \cdot 10^6$. |
|---|---|---|---|
| T = 24·83° C. | | | |
| 0·97 | 0·89 | 1·0005 | 187 |
| 25·7 | 23·7 | 1·0130 | 182 |
| 49·6 | 45·7 | 1·0262 | 190 |
| 71·3 | 65·0 | 1·0379 | 192 |
| 96·9 | 88·4 | 1·0517 | 191 |
| 118·0 | 107·0 | 1·0628 | 192 |
| 123·4 | 111·7 | 1·0653 | 191 |
| 118·2 | 107·1 | 1·0627 | 191 |
| 97·3 | 88·8 | 1·0518 | 191 |
| 72·4 | 66·5 | 1·0384 | 190 |
| 49·1 | 45·2 | 1·0257 | 188 |
| 25·1 | 23·1 | 1·0125 | 180 |

Table I. (*cont.*).

| $P_{Atm.}$ | $d_{Amagat.}$ | $\epsilon$. | $\frac{\epsilon-1}{\epsilon+2} \cdot \frac{1}{d} . 10^6$. |
|---|---|---|---|
| T = 75·03° C. | | | |
| 0·97 | 0·76 | 1·0004 | 175 |
| 26·05 | 20·3 | 1·0114 | 186 |
| 48·6 | 37·8 | 1·0217 | 190 |
| 73·6 | 56·7 | 1·0330 | 192 |
| 97·8 | 74·7 | 1·0432 | 190 |
| 118·3 | 89·6 | 1·0526 | 193 |
| 131·8 | 99·2 | 1·0578 | 190 |
| 120·8 | 91·4 | 1·0536 | 192 |
| 97·8 | 74·7 | 1·0433 | 190 |
| 73·8 | 56·9 | 1·0330 | 191 |
| 49·3 | 38·3 | 1·0218 | 188 |
| 25·3 | 19·8 | 1·0106 | 179 |
| T = 75·04° C. | | | |
| 135·8 | 102·1 | 1·0597 | 191 |
| 120·3 | 91 0 | 1·0533 | 192 |
| 97·7 | 74·6 | 1·0435 | 191 |
| 73·3 | 56·5 | 1·0325 | 190 |
| 50·7 | 39·4 | 1·0225 | 189 |
| 26·0 | 20·3 | 1·0111 | 182 |
| 0·97 | 0·76 | 1·0004 | 175 |
| 23·1 | 18·1 | 1·0100 | 185 |
| 53·1 | 41·2 | 1·0236 | 189 |
| 73·1 | 56·4 | 1·0328 | 192 |
| 96·1 | 73·5 | 1·0424 | 190 |
| 117·5 | 89·0 | 1·0517 | 190 |
| 133·7 | 100·6 | 1·0586 | 190 |
| T = 125·18° C. | | | |
| 146·2 | 92·2 | 1·0558 | 193 |
| 122·0 | 79·8 | 1·0469 | 193 |
| 97·0 | 64·2 | 1·0377 | 193 |
| 71·2 | 47·6 | 1·0275 | 191 |
| 50·1 | 33·8 | 1·0192 | 188 |
| 25·7 | 17·5 | 1·0095 | 181 |
| 0·97 | 0·66 | 1·0003 | 176 |
| 25·4 | 17·2 | 1·0089 | 172 |
| 49·3 | 33·2 | 1·0185 | 184 |
| 74·8 | 50·0 | 1·0285 | 188 |
| 97·2 | 64·3 | 1·0369 | 189 |
| 121·0 | 79·2 | 1·0456 | 189 |
| 132·1 | 86·0 | 1·0496 | 189 |

From the experimental figures the value of the Clausius-Mosotti expression $\left(\frac{\epsilon-1}{\epsilon+2}\cdot\frac{1}{d}\right)$ was calculated. The values of the density at 25° and 75° were obtained from the critical summary of Bartlett's isotherm data given by Deming and

Fig. 1.




Shupe[4]. The values at 125° were obtained from the series expression of the nitrogen isotherms given by Holborn and Otto[5].

With the exception of the points at 1 and 25 atm. (where the small value of $(\epsilon-1)$ made the measurements less accurate) the value of the Clausius-Mosotti expression is seen to be independent of pressure and temperature. Using the

average value of the constant ($191 \times 10^{-6}$) the value of the dielectric constant at 1 atm. and 0° was calculated to be 1·000573, which compares favourably with the figures given in the literature.

TABLE II.

| | |
|---|---|
| Tangl[2] ..... | $\epsilon = 1{\cdot}000581$ |
| Bodareu[2] ..... | 1·000587 |
| Zahn[1] ..... | 1·000581 |
| Broxon[2] ..... | 1·000590 |
| Our value ..... | 1·000573 |

The work is being continued to higher pressures and a pressure balance used for measurement.

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(1) C. T. Zahn, Phys. Rev. xxiv. p. 400 (1924) ; xxvii. p. 455 (1926).
(2) K. Tangl, *Ann. der Physik*, xxvi. p. 59 (1908). E. Bodareu, *Atti Acad. Lincei*, xxii. (2) p. 480 (1913). J. W. Broxon, Phys. Rev. xxxviii. p. 2049 (1931).
(3) R. T. Lattey and O. Gatty, Phil. Mag. (7) vii. p. 985 (1929).
(4) E. Deming and Lola E. Shupe, Phys. Rev. xxxvii. p. 638 (1931).
(5) L. Holborn und J. Otto, *Zeit. f. Phys.* xxiii. p. 77 (1924) ; xxx. p. 320 (1924). Wien-Harms, *Handb. d. Exp. Phys.* viii. (2) pp. 167, 168.

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## CXIV. *On the Superconductivity of Alloys containing Gold and Silver.* *By* Prof. J. C. McLennan, *F.R.S.*, J. F. Allen, *M.A.**, *and* J. O. Wilhelm, *M.A.*†

IN two earlier papers[1] published by us on the appearance of superconductivity in alloys, particular attention was paid to the superconductive elements Pb, Sn, and Tl when alloyed with the non-superconductive members of the fifth group, namely, Bi, Sb, As, and P. It was pointed out at that time, that these alloys in general exhibited higher transition temperatures than did the pure superconductive metals alone.

The present publication is the third of the series, and deals with observations made on the alloy systems Ag–Sn, Au–Sn, and Au–Pb. Our attention was drawn to these groups by the work of de Haas and his associates. De Haas[2] found

* Student of the National Research Council of Canada.
† Communicated by the Authors.

that the components $Ag_3Sn$ and AuSn were non-superconductive ; while in a previous investigation by us, the eutectics of Ag–Pb and Au–Pb were found to become superconducting at temperatures very close to the transition point of pure lead. It appeared, therefore, that the effect produced on the superconductivity by Au and Ag in Pb differed markedly from that produced in Sn.

This difference was considered worthy of our closer investigation. A comparison of the equilibrium diagrams of Ag–Sn, Au–Sn, Ag–Pb, and Au–Pb, as determined by Petrenko [3] and Vogel [4], showed that the system Ag–Pb possessed a simple two branch melting-point curve with one eutectic, whereas the other three systems exhibited more complex structures, forming compounds and containing several eutectics. The system Ag–Sn possesses the compound $Ag_3Sn$ ; the Au–Sn system has AuSn, $AuSn_2$, and $AuSn_4$, while in the Au–Pb system $Au_2Pb$ and $AuPb_2$ are present. An examination was made of all these compounds and of the various eutectics in the three systems.

The alloys were prepared in the usual manner, being melted together, strained through glass wool, and cast in pyrex glass tubes *in vacuo*. The casting took place in a thin section of the tube which had been drawn out to a length of about five centimetres. As a precaution against breakage, some of the alloys being crystalline and very brittle, the glass was cracked away only from the ends of the sample and the rod was thus left almost completely encased in the thin glass tube. To the ends of the sample were soldered current and potential leads of fine copper wire ; the potential drop along the rod was measured on a Cambridge Instrument Co. Vernier Potentiometer.

### *Description of the Alloys.*

(1) *The Ag–Sn System.*

(*a*) Ag–Sn (Eutectic).

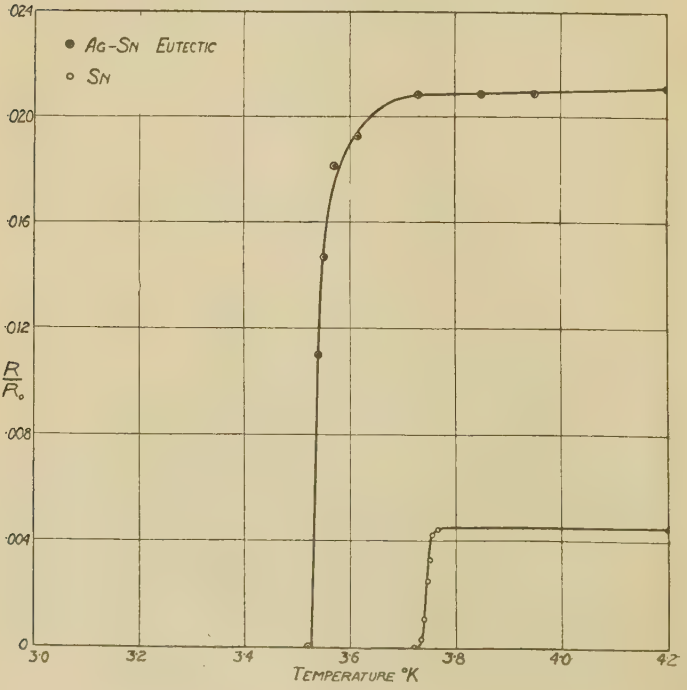
This alloy, which is the main eutectic of the system, contains 4 per cent. Ag by weight. According to Vogel, the eutectic constituents are Sn and $Ag_3Sn$. The measurements made on this sample are given in Table I. and shown graphically in fig. 1 in comparison with the superconducting curve for pure Sn. It will be seen that for this alloy the residual resistance above the transition point has increased about fivefold and that the superconducting point is about $\cdot 2^\circ$ lower than that for pure Sn.

TABLE I.

Ag–Sn (Eutectic).

| T° K. | R (microhms). | $R/R_0$. |
|---|---|---|
| 300° | 15200 | 1·113 |
| 273 | 13650 | 1·000 |
| 85 | 3290 | 0·2402 |
| 4·2 | 288 | ·0211 |
| 3·95 | 286 | ·02095 |
| 3·85 | 285 | ·02087 |
| 3·73 | 285 | ·02087 |
| 3·615 | 263 | ·01927 |
| 3·57 | 230 | ·01818 |
| 3·55 | 200 | ·01465 |
| 3·54 | 150 | ·01100 |
| 3·52 | 0 | ·0000 |

Fig. 1.



(*b*) $Ag_3Sn$–Sn (Eutectic).

In the equilibrium diagram, this alloy, which contains 50 per cent. Ag, is given as a minor eutectic. The constituents

Fig. 2.

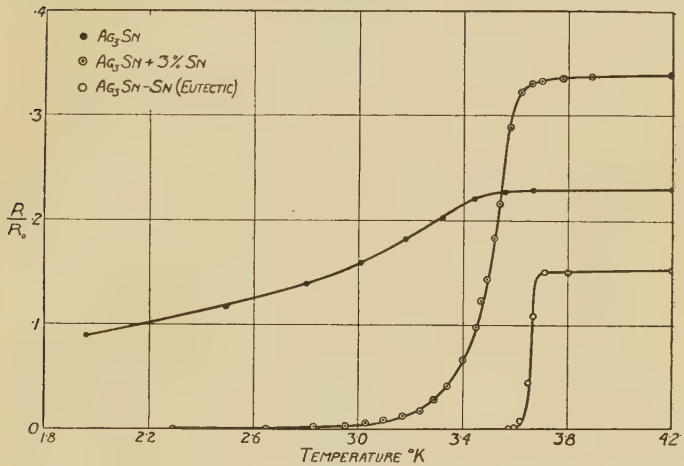



Table II.

$Ag_3Sn$–Sn (Eutectic).

| T° K. | R (microhms). | $R/R_0$. |
|---|---|---|
| 300 | 1900 | 1·071 |
| 273 | 1775 | 1·000 |
| 85 | 700 | 0·394 |
| 10 | 275 | ·155 |
| 4·2 | 270 | ·1521 |
| 3·8 | 267 | ·1504 |
| 3·71 | 267 | ·1504 |
| 3·67 | 193 | ·1087 |
| 3·65 | 79 | ·0445 |
| 3·62 | 13 | ·0073 |
| 3·60 | 2 | ·001113 |
| 3·57 | 0 | ·0000 |

of the alloy are $Ag_3Sn$ and the eutectic (*a*) above. It is seen from fig. 2 and Table II. that the superconducting point for this alloy is very slightly higher than that for (*a*) above.

(*c*) $Ag_3Sn$.

This compound, which contains 73 per cent. Ag, has previously been examined by de Haas, van Aubel, and Voogd, who found it to be non-superconducting, although it did show a considerable drop in resistance below 3·6° K. Our measurements made on this alloy (fig. 2 and Table III.) agree very closely with those made by de Haas.

TABLE III.

$Ag_3Sn$.

| T° K. | R (microhms). | $R/R_0$. |
|---|---|---|
| 300 | 9740 | 1·080 |
| 273 | 9020 | 1·000 |
| 85 | 4050 | 0·449 |
| 8 | 2101 | ·2342 |
| 4·2 | 2060 | ·2285 |
| 3·67 | 2060 | ·2285 |
| 3·56 | 2049 | ·2272 |
| 3·44 | 1985 | ·2200 |
| 3·32 | 1826 | ·2024 |
| 3·18 | 1630 | ·1810 |
| 3·01 | 1434 | ·1588 |
| 2·80 | 1250 | ·1387 |
| 2·49 | 1050 | ·1163 |
| 1·96 | 800 | ·0888 |

(*d*) $Ag_3Sn$ +3 per cent. Sn.

Due to a mishap in the preparation of a sample of $Ag_3Sn$, a small portion of the Ag was not dissolved in the melt. The resulting alloy therefore contained a slight excess of Sn. Nevertheless, measurements made on the alloy (Table IV. and fig. 2) are of interest, since they show the nature of the change from $Ag_3Sn$–Sn (Eutectic) to pure $Ag_3Sn$ as the percentage of Ag increases. In the curve for (*d*) some of the features of the curves for both (*b*) and (*c*) are apparent. We have the very steep upper part of the transition curve which is similar to that for (*b*); we have also, as in the curve for (*c*), a slow final approach to superconductivity. In the case of (*c*), however, superconductivity disappears entirely.

(2) *The Au–Sn System.*

(*e*) $AuSn_4$–Sn (Eutectic).

The proportion of Au here is 9 per cent. by weight, and the alloy constitutes the main eutectic of the Au-Sn system.

TABLE IV.

$Ag_3Sn$ + 3 per cent. Sn.

| T° K. | R (microhms). | $R/R_0$. |
|---|---|---|
| 300 | 4737 | 1·064 |
| 273 | 4450 | 1·000 |
| 85 | 2484 | 0·559 |
| 8·0 | 1515 | ·342 |
| 4·2 | 1500 | ·3373 |
| 3·9 | 1497 | ·3365 |
| 3·78 | 1494 | ·3360 |
| 3·70 | 1488 | ·3348 |
| 3·66 | 1470 | ·3330 |
| 3·62 | 1437 | ·3230 |
| 3·58 | 1287 | ·2895 |
| 3·54 | 960 | ·2155 |
| 3·52 | 816 | ·1833 |
| 3·49 | 630 | ·1415 |
| 3·47 | 540 | ·1212 |
| 3·45 | 429 | ·0965 |
| 3·40 | 294 | ·0662 |
| 3·34 | 189 | ·0425 |
| 3·29 | 120 | ·0270 |
| 3·24 | 84 | ·0189 |
| 3·17 | 63 | ·0142 |
| 3·10 | 45 | ·0101 |
| 3·03 | 33 | ·00742 |
| 2·95 | 21 | ·00472 |
| 2·86 | 15 | ·00338 |
| 2·83 | 10·5 | ·00236 |
| 2·65 | 6 | ·00135 |
| 2·30 | 0 | ·0000 |

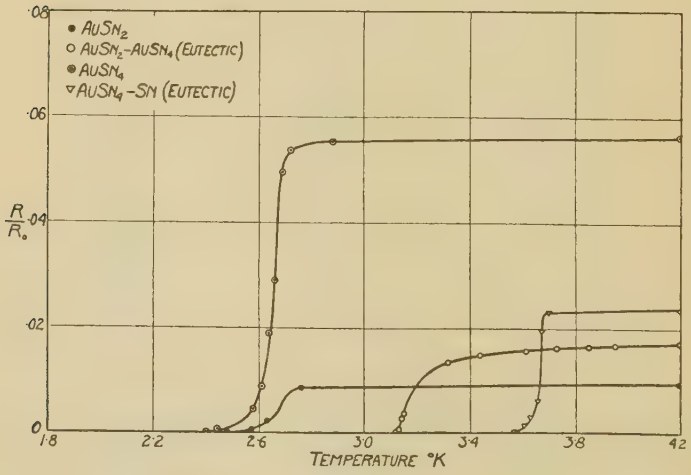
The data obtained (Table V. and fig. 3) show that this alloy has the highest superconducting temperature of the system, the point falling about ·2° below that for pure Sn.

TABLE V.

Au–$Sn_4Sn$ (Eutectic).

| T° K. | R (microhms). | $R/R_0$. |
|---|---|---|
| 300 | 4830 | 1·098 |
| 273 | 4400 | 1·000 |
| 85 | 1210 | 0·275 |
| 7·0 | 103 | ·0234 |
| 4·2 | 102 | ·0232 |
| 3·7 | 101 | ·02295 |
| 3·68 | 85 | ·01933 |
| 3·66 | 27 | ·00614 |
| 3·63 | 15 | ·00341 |
| 3·61 | 8 | ·00182 |
| 3·57 | 0 | ·0000 |

Fig. 3.




(*f*) $AuSn_4$.

This compound contains 29 per cent. Au. The measurements are given in Table VI. and shown in fig. 3. The compound becomes superconducting at 2·4° K.

(*g*) $AuSn_2$–$AuSn_4$ (Eutectic).

The eutectic between these two compounds contains 40 per cent. Au, and the measurements (Table VII. and fig. 3) show

that it has a much higher superconducting point than that for $AuSn_4$ ; the transition occurs at 3·125° K.

(*h*) $AuSn_2$.

This compound contains 45 per cent. Au. The measurements obtained are given in Table VIII. and shown graphically in fig. 3. The curve is very similar to that for $AuSn_4$. Superconductivity appears at 2·48° K.

TABLE VI.

$AuSn_4$.

| T° K. | R (microhms). | $R/R_0$. |
|---|---|---|
| 300 | 11400 | 1·095 |
| 273 | 10350 | 1·000 |
| 85 | 4125 | 0·399 |
| 7·5 | 603 | ·0582 |
| 4·2 | 582 | ·0562 |
| 2 88 | 570 | ·0551 |
| 2·72 | 555 | ·0536 |
| 2 69 | 510 | ·04925 |
| 2·66 | 300 | ·02899 |
| 2·64 | 195 | ·01885 |
| 2·61 | 90 | ·0087 |
| 2·58 | 42 | ·00405 |
| 2·44 | 6 | ·000570 |
| 2·4 | 0 | ·0000 |

(*i*) AuSn.

De Haas and his associates have measured this alloy and announce that it is non-superconductive.

(*j*) AuSn–Au (Eutectic).

It was thought that perhaps this eutectic, which contains 80 per cent. Au, might become superconducting within the temperature range of our apparatus. The alloy, however, retained a comparatively high resistance down to 1·96 K., as shown in Table IX.

(3) *The Au–Pb System.*

(*k*) $AuPb_2$–Pb (Eutectic).

This alloy, containing 15 per cent. Au, is the main eutectic of the system Au–Pb. Measurements which we made on

this alloy were published some time ago as one of a group of lead alloys. The curve obtained has been reproduced in fig. 4

Table VII.

$AuSn_2$–$AuSn_4$ (Eutectic).

| T° K. | R (microhms). | $R/R_0$. |
|---|---|---|
| 300 | 14570 | 1·071 |
| 273 | 13600 | 1·000 |
| 85 | 4560 | 0·335 |
| 4·2 | 230 | ·0169 |
| 3·95 | 224 | ·01646 |
| 3·85 | 222 | ·01632 |
| 3·73 | 218 | ·01602 |
| 3·615 | 213 | ·01567 |
| 3·44 | 200 | ·01470 |
| 3·32 | 181 | ·01330 |
| 3·15 | 50 | ·00368 |
| 3·14 | 40 | ·00294 |
| 3·13 | 10 | ·000735 |
| 3·125 | 0 | ·0000 |

Table VIII.

$AuSn_2$.

| T° K. | R (microhms). | $R/R_0$. |
|---|---|---|
| 300 | 3230 | 1·133 |
| 273 | 2850 | 1·000 |
| 85 | 819 | 0·2875 |
| 11 | 37 | ·01296 |
| 4·2 | 26 | ·00913 |
| 2·76 | 24 | ·00843 |
| 2·63 | 6 | ·00210 |
| 2·57 | 2 | ·0007 |
| 2·48 | 0 | ·0000 |

for the purpose of comparison with the present results from other alloys of the system. This alloy becomes superconducting at 7·0° K.

(*l*) $AuPb_2$.

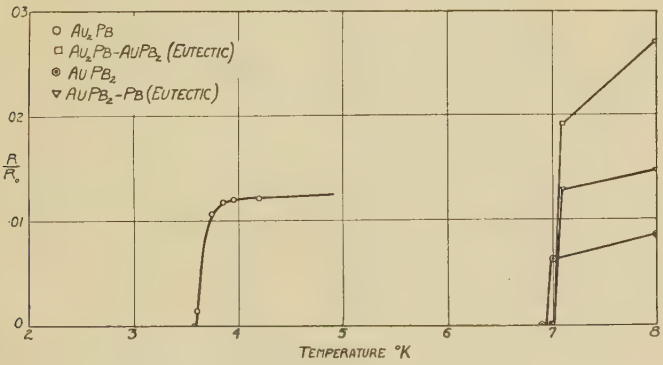
The gold-content of this compound is 33 per cent. Au by weight. Measurements made on it are shown graphically in fig. 4 from data in Table X. The superconducting point 6·9° K. is very little lower than that for the eutectic (*k*).

TABLE IX.

AuSn–Au (Eutectic).

| T° K. | R (microhms). | $R/R_0$. |
|---|---|---|
| 300 | 8600 | 1·100 |
| 273 | 7820 | 1·000 |
| 85 | 2820 | 0·3605 |
| 4·2 | 1280 | ·1638 |
| 1·96 | 1280 | ·1638 |

Fig. 4.




(*m*) $Au_2Pb$–$AuPb_2$ (Eutectic).

This eutectic contains 55 per cent. Au and becomes superconducting at the same temperature as eutectic (*k*), namely, 7·0° K. The data are given in Table XI. and shown in fig. 4.

(*n*) $Au_2Pb$.

This is a very crystalline compound with a rhombic structure, according to Vogel. The compound contains 65 per cent. Au and shows a transition temperature 3·59° K. (Table XII. and fig. 4), which is much lower than that for any of the other alloys in the AuPb system.

TABLE X.

$AuPb_2$.

| T° K. | R (microhms). | $R/R_0$. |
|---|---|---|
| 300 | 10470 | 1·085 |
| 273 | 9590 | 1·000 |
| 85 | 2990 | 0·3115 |
| 8·0 | 82 | ·00855 |
| 7·0 | 60 | ·00626 |
| 6·9 | 0 | ·0000 |

TABLE XI.

$Au_2Pb$–$AuPb_2$ (Eutectic).

| T° K. | R (microhms). | $R/R_0$. |
|---|---|---|
| 300 | 4070 | 1·102 |
| 273 | 3705 | 1·000 |
| 85 | 1160 | 0·3108 |
| 8·0 | 84 | ·0270 |
| 7·1 | 71 | ·01919 |
| 7·0 | 0 | ·0000 |

TABLE XII.

$Au_2Pb$.

| T° K. | R (microhms). | $R/R_0$. |
|---|---|---|
| 300 | 9800 | 1·090 |
| 273 | 9000 | 1·000 |
| 85 | 2600 | 0·291 |
| 4·2 | 110 | ·01222 |
| 3·95 | 108 | ·01200 |
| 3·85 | 106 | ·01177 |
| 3·73 | 104 | ·01056 |
| 3·615 | 13 | ·001445 |
| 3·59 | 0 | ·0000 |

*Discussion.*

In the foregoing results, we immediately notice two outstanding features. First, in alloys with the superconductive elements, that Au and Ag produce an effect on the transition temperature which is the opposite of that produced by Bi, Sb, and As. When we are observing alloys containing the latter metals, we find, usually, a pronounced elevation of

the superconducting point, while in alloys with Au and Ag we find an equally pronounced depression of the superconducting temperature. Second, that a binary alloy system composed of one superconductor and one non-superconductor does not necessarily have a unique transition temperature.

On closer examination of the system Ag–Sn, we see that the two eutectics became superconducting at temperatures slightly below that for pure tin, while the compound $Ag_3Sn$ has almost entirely lost the property of superconductivity.

We are thus led directly to infer that eutectics are more "stable" as regards superconductivity than are compounds, or, more generally, than are proportions other than eutectics. The latter generalization is based on the case of $Ag_3Sn$ + 3 per cent. Sn, which, as has been stated before, occurs between $Ag_3Sn$ and $Ag_3Sn$–Sn (Eutectic). Here we see that the addition of 20 per cent. Ag to the $Ag_3Sn$–Sn eutectic causes a lowering of the transition temperature by 1·3°, while the further addition of 3 per cent. Ag results in the complete disappearance of superconductivity. In dealing with this system, it is also interesting to note that Ag–Sn (Eutectic), containing only 4 per cent. Ag, has a slightly lower superconducting point than has $Ag_3Sn$–Sn (Eutectic), which contains 50 per cent. Ag. Considering the rather large proportion of the non-superconductive Ag in the latter alloy, the result is somewhat unexpected.

The system Ag–Sn furnishes also a striking example of the variation of the residual resistance as the change in proportion takes place from pure tin, through the various alloys, to pure silver. This is shown in the following table :—

| Alloy. | Residual resistance ($R/R_0$). |
|---|---|
| Pure Sn ..... | ·0045 |
| Ag–Sn (Eut ) 4 per cent. Ag ..... | ·021 |
| $Ag_3Sn$–Sn (Eut.) 50 per cent. Ag . | ·15 |
| $Ag_3Sn$+3 per cent. Sn (70 per cent. Ag). | ·336 |
| $Ag_3Sn$ (73 per cent. Ag) ..... | ·228 |
| Pure Ag ..... ..... | ·006 |

In examining the system Au–Sn, we see the inference drawn from the Ag–Sn system (*i. e.*, that eutectics possess higher superconducting points than compounds) to be demonstrated more conclusively. The transition temperatures for the two compounds $AuSn_2$ and $AuSn_4$ are very close together, being 2·48° and 2·4° respectively. The eutectic $AuSn_2$–$AuSn_4$ shows a much higher transition temperature

than either of its component compounds, and the eutectic $AuSn_4$–Sn, which contains 91 per cent. Sn, shows a superconducting point very close to that for pure Sn. Soon after the gold-content increases past the $AuSn_2$ proportion, the superconductivity disappears, since it is absent both in the compound AuSn and in the eutectic AuSn–Au. In the system Au–Sn we notice the same unusual inversion of superconducting points that appeared in the Ag-Sn system, but in the case of Au–Sn, the inversion takes place between the compounds. Here we see that the compound $AuSn_2$ has a higher transition temperature than $AuSn_4$, which is the richer in tin.

Our primary inference concerning the greater superconductive stability of eutectics over compounds still holds in the case of the Au–Pb system. We have in this system two compounds, $Au_2Pb$ and $AuPb_2$, and two eutectics. Both eutectics become superconducting at 7·0° K. (the transition temperature of lead is 7·2° K.) and the compound $AuPb_2$, only slightly lower, at 6·9° K. $Au_2Pb$, on the other hand, appears to be totally different from the rest of the system, since its transition temperature is depressed 3·6° below that of lead. Besides differing so greatly from the others with regard to its superconducting point, this compound $Au_2Pb$ differs also in certain other physical properties. For example, it is a very highly crystallized substance and can be quite easily pulverized, whereas the other alloys of the system are somewhat malleable. It seems, therefore, that the amount of gold must increase to over 60 per cent. of the alloy before the superconductivity is affected to any degree. This lack of effect on superconductivity in alloys with low gold-content is also clearly shown in the shape of the resistance curve immediately above the transition region of the temperature. The steeply sloping resistance curve in this temperature region is characteristic of pure lead.

In all the cases examined we have seen that superconductivity appears at higher temperatures in eutectic alloys than in compounds of the same metals. That is to say, more specifically, that in the case of a eutectic between compounds, the resulting superconductivity point is raised; but where the eutectic occurs between a compound and a pure superconductor, the resulting transition temperature lies somewhere between the superconducting points of the two components. The reason for this becomes a little clearer when we examine the differences in structure between compounds and eutectics. Compounds are microscopically homogeneous, whereas eutectics are not microscopically but only

macroscopically homogeneous. Eutectics are composed usually of mixtures of crystals of the components of the alloy, the components being sometimes compounds, or, more usually, solid solutions of one metal in the other. De Haas[5] has shown that in multicrystal specimens of tin, superconductivity begins to appear at a higher temperature than in single crystals. He has also found that tensional stresses on a sample of tin result in the onset of superconductivity at a higher temperature than normal. He therefore assumed that tensional stresses must exist between adjacent crystal faces of the multicrystal specimen and that superconductivity appears first in these places.

The present investigation bears out de Haas's theory: that is, between the crystal faces in the mixed structure of a eutectic there must exist forces of considerable magnitude which cause superconductivity to appear at temperatures other than would be expected. That these internal forces must be of greater magnitude in eutectics than in compounds or alloys of proportions other than eutectic, is essential from the nature of the eutectic. Since the eutectic is by definition the proportion with the lowest melting-point, it must therefore be, according to the theory of change of state, the proportion in which the greatest internal forces are present.

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## CXV. *Notices respecting New Books.*

*Foundations of Mathematics.* By F. P. Ramsey. (Kegan Paul. 1931. Price 15*s.*)

A volume from the pen of one able in both the mathematical and philosophical fields is always of first-rate importance, and the posthumous collection of the writings of F. P. Ramsey is accordingly most welcome; for the realm of Mathematical Philosophy is one in which the mathematician is too often at a loss to appreciate philosophical difficulty, and the philosopher is at a loss to grasp the technical demands of the mathematician. Ramsey was one of the very few who suffer from neither deficiency, and he may well be said to have acquired a reputation both as a

philosophical thinker and as a mathematician. Although the subject-matter varies considerably as we proceed through his papers, yet there is a perceptible continuity of thought which runs through the volume and gives a clear indication of the author's mode of thinking. At the same time the influence of Wittgenstein is very much in evidence, while a great deal of the work may be regarded as correction and adjustment of the work of Russell. In consequence a familiarity with the writings of these authors is desirable when reading almost any portion of Ramsey's work, and in places it is positively essential.

The first paper, from which the volume derives its name, is devoted to the criticism and development of the theory of Classes and Cardinals and of the Arithmetical Continuum, as set forth by Russell and Whitehead in their 'Principia Mathematica.' By means of a new discussion of "functions" and a new associated theory of types the difficulties which arise in connexion with the Russell theory of types are removed, and the Axiom of Reducibility is no longer necessary as an axiom. This paper is undoubtedly a valuable contribution to the theory of Mathematical Philosophy. It is followed by a more popularly constructed paper entitled "Mathematical Logic," consisting most pleasantly of a sketch of the ideas of Weyl, Brouwer, Hilbert, and Wittgenstein.

After a paper on a particular problem of formal logic there follow two papers entitled "Universals" and "Facts and Propositions." The former is a criticism of Russell's view that universals and particulars are essentially distinct; Russell's arguments in this connexion are shown to be inadequate, and it would seem that Ramsey held that the distinction was a fiction of words. The latter is a discussion of the nature of belief.

The last of the previously published papers included in this volume is called "Truth and Probability." It attempts to give an account of the principles underlying what is termed partial belief, and if it does not develop to any great extent, it at least has more contact with the probability judgments of human beings than the mathematical theory of probability, as set forth by Poincaré, for example. The volume concludes with a number of sketches of theories, as jotted down by the author and left in comparatively undeveloped states.

Throughout the volume it is of interest and importance to observe the way in which language and symbolism are presented as possible sources of apparent, but resolvable, difficulty. This may be regarded as following Wittgenstein, and it is certainly true that mathematicians have long been of this opinion in connexion with the difficulties of the theory of types, though their reasons for that opinion have been inadequate. But one wonders when Ramsey explains how he avoids certain confusions whether his words and sentences have one and only one meaning. Be that as it may, he has pointed out certain blemishes in other writers, he has smoothed out the difficulties of a profound theory—these are achievements in themselves.

*Scientific Inference.* By HAROLD JEFFREYS, M.A., D.Sc., F.R.S. [Pp. vii+247.] (Cambridge University Press, 1931. Price 10*s.* 6*d.*)

THE problem of scientific truth is one which is of fundamental importance, not only in its wider philosophical setting but also from a purely practical pragmatic standpoint. What is meant by the assertion that a scientific proposition is true? That is the main (though not the only) question that Dr. Jeffreys tries to answer in this book. The book will be disappointing to those who expect too much; they might hope to find justification for the choice of the simpler laws, where there are alternatives, and a sure foundation for the belief that it is possible to learn from experience. That hope is soon removed, for it is taken as a primitive postulate that it is possible to make inferences beyond the data known directly by sensation, and a high prior probability is attributed, as a necessary assumption, to simple laws. On that basis Dr. Jeffreys is able to show, by a most elegant application of the theory of probability, that the posterior probability of a possible simple law, after suitable tests have been made, may become extremely high. The views here developed had their beginnings some years ago in a series of papers in the 'Philosophical Magazine' by the author and Dr. Dorothy Wrinch. It is a considerable step that certain aspects of scientific inference have in this way been treated virtually in a quantitative manner; and in so far as the treatment is successful the essential problem has been narrowed down to an examination of the assumed postulates which form the starting-point of the deductive argument.

Relevant to the theme are discussions of probability, sampling, quantitative laws, errors, and physical magnitudes. Chapters on mensuration, Newtonian dynamics, and light and relativity unified only the general point of view of the author, have a value rather as separate essays, being succinct outlines of a systematic treatment of these subjects, with frequent novel methods of presentation. There follows a discussion of miscellaneous topics, such as ultimate concepts, and an eclectic account of some other theories of scientific knowledge.

The book as a whole seems to suffer from a lack of artistic unity; but there is a compensation, in that the inevitably difficult matter in some sections is balanced by the pleasant manner of others. The book might perhaps more appropriately have been called "Essays on Scientific Inference and other Topics." As such, it should be welcomed by a wide circle of readers, whether their main interests are mathematical, scientific, or philosophical.

*Cartesian Tensors.* By H. JEFFREYS. (Cambridge University Press, 1931. Price 5*s.* net.)

PERHAPS one of the most useful consequences of the Theory of Relativity is the fact that mathematicians and physicists have become accustomed to tensors. If we add the Einstein summation convention, we have a powerful notation of extraordinary

compactness. Part, indeed, of the pleasure that one derives from the study of modern physical and mechanical developments is due to the elegance of the mathematical reasoning. Unfortunately, however, a rumour has been spread with remarkable persistence to the effect that tensors are very difficult to understand, whereas they are in reality no more difficult than most of the subjects that a good honours student at our universities can readily assimilate. Dr. Jeffreys has therefore done considerable service to mathematicians and physicists by the publication of his book on 'Cartesian Tensors.' His object is to illustrate the use of tensor methods in Solid Geometry, Particle and Rigid Dynamics, Statics, Hydrostatics, Hydrodynamics, Elasticity, as well as the general theory of Electric and Magnetic Fields. Dr. Jeffreys does not claim to give a complete theory of these subjects. His aim is rather to illustrate the value of tensor notation and methods by reference to the fundamentals of these subjects. The results are in many cases interesting and striking, and there can be no doubt that a student who has received lectures on these matters based upon traditional methods will find Dr. Jeffreys's treatment very stimulating. Thus, the fact that Love's strain components do not constitute a tensor until some of them are halved, or that, in three dimensions, any vector can be associated with an anti-symmetrical tensor of the second order, shed much light upon the theory of continuous media.

The treatment of tensors in this book is necessarily incomplete. As the author restricts himself to rectangular Cartesian axes, the distinction between covariant and contravariant tensors disappears completely. This is perhaps a temporary advantage, but it means that the student can have only a very faint impression of the real beauty of tensor calculus. Would not the author have done better by showing how rectangular Cartesians lead to this simplification, by including some extension to other types of coordinates that are familiar to the student, and of even greater importance in many branches of physics and mechanics?

A particularly valuable feature of the book is the brief study of the substitution tensor, the alternating tensor, and isotropic tensors. The proof that there is no isotropic tensor of the first order other than zero, of the second order other than the substitution tensor, of the third order other than the alternating tensor, and also the form assumed by the general isotropic tensor of the fourth order are very instructive and illuminating. The treatment would probably have been even more instructive if isotropic tensors had been included in the first chapter of the book.

Dr. Jeffreys's latest production is a valuable addition to mathematical literature; yet it somehow leaves one with a haunting fear of the danger involved in the very simplifications of tensor methods that are characteristic of the book. Is it not dangerous to tell a student without some qualification that the gradient of a vector is a tensor of the second order?

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

# INDEX TO VOL. XIII.

ABSORPTION and transmission of water by porous bodies, 632.
—— of X-rays in gases and vapours, 505.
Accommodation coefficient of liquid and solid surfaces, the effect of intensive drying on, 1014.
Activities and the standard state: activity coefficients, 283; electrode potentials, 291.
Aerosols, homogeneous, the theory of coagulation of, 523.
——, theory of the electrification of; correction, 736.
Airscrew, sound generated by a rotating, 99.
Alexander (F. W.) on the decay of torsional oscillations of an iron wire, effect of variation of period, 934.
Allen (J. F.) McLennan (Prof. J. C.), and Wilhelm (J. O.) on the superconductivity of alloys containing gold and silver, 1196.
Aluminium rectifying cell, characteristic curves of, 76.
Anderson (W.) on the dielectric constant and power factor of some solid dielectrics at radio frequencies, 986.
Anomalous variation of the electrical conductivity of quartz, with temperature, 907.
Appleyard (Dr. E. T. S.) on an attempt to detect high photoelectric absorption in cæsium vapour at double the series limit, 300.
Arc, spark, and glow: a note on nomenclature, 824.
Awbery (J. H.) on the current flow in a circular disk, 674.
Backhurst (Ivor) on the existence of the J-phenomena, 28.
Bailey (Prof. V. A.) on light absorption, the Raman effect, and the motions of electrons in gases, 993.
Band spectra of boron fluoride, 501.
Banerji (D.) and Ganguli (R.) on the distribution of space-potential in striated and other forms of high-frequency discharge, 494.
Banerji (S. K.) on Oberbeck's vortices, 865.
Barnes (L. L.) on the characteristic curves of the aluminium rectifying cell, 76.
Bates (Dr. L. F.) on the thermoelectric properties of ferromagnetic substances, 393.
Beams and struts, the more accurate calculation of the deflexions of, 310.
Bending of columns of varying cross-section, 845.
Benzene, variations in the refractive index during intensive drying, 249.
Berry (G.) and Kimball (W. S.) on entropy, strain, and the Pauli exclusion principle, 1131.
Bickley (Dr. W. G.) on a simple method for the numerical solution of differential equations, 1006.
Biswas (S. C.) on the Raman spectra in liquid and gaseous methane, 455.
Books, new:—Sir Oliver Lodge's Past Years: an Autobiography, 197; Mr. C. P. Smyth's Dielectric Constant and Molecular Structure, 199; Mr. B. B. Low's Mathematics, 199; Dr. H. Jeffreys's Operational Methods in Mathematical Physics, 200; James Clerk Maxwell: a Commemoration Volume, 1831–1931, 200; Sir James Jeans's The Mysterious Universe, 200; Mons.

Pierre Brunet's L'Introduction des Théories de Newton en France au XVIII^e siècle, 322; Dr. F. W. Mellor's A Comprehensive Treatise upon Inorganic and Theoretical Chemistry, Vol. IX., 323; Prof. L. B. Loeb's The Nature of a Gas, 324; Dr. R. G. J. Fraser's Molecular Rays, 324; Dr. W. N. Bond's Numerical Examples in Physics, 325; Drs. G. W. Stewart and R. B. Lindsay's Acoustics: a Text on Theory and Application, 325; Dr. B. A. Keen's The Physical Properties of the Soil, 527; Prof. R. von Mises's Wahrscheinlichkeitsrechnung und ihre Anwendung in der Statistik und Theoretischen Physik, 528; Mons. G. Ribaud's Traité de Pyrométric Optique, 529; Prof. M. Siegbahn's Spektroskopie der Röntgenstrahlen, 530; Drs. H. C. Levinson and E. B. Zeisler's The Law of Gravitation in Relativity, 531; Mr. F. C. McKerrow's Novius Organum, 531; Handbuch der Experimentalphysik, Hydro- und Aerodynamik, Strömungslehre und Allgemeine Versuchstechnik, 532; Prof. L. Brillouin's Die Quantenstatistik und ihre Anwendung auf die Elektronentheorie der Metalle, 737; Mons. J. P. Pomey's Cours d'Électricité Théorique, 738: Mr. P. Vigoureux's Quartz Resonators and Oscillators, 738; Prof. H. A. Lorentz's Die Maxwellsche Theorie, 739; Prof. P. Debye's The Dipole Moment and Chemical Structure, 740; Drs. E. F. and K F. Armstrong's Monograph on Biochemistry: the Glycosides, 869; Dr. V. F. Lenzen's The Nature of Physical Theory, a Study in Theory of Knowledge, 870; Prof. W. Wilson's Theoretical Physics, Mechanics and Heat, 871; Prof. Conway and Dr. Synge's Mathematical Papers of Sir William Rowan Hamilton; Geometrical Optics, 1075; Prof. A. Harden's Alcoholic Fermentation, 1077; Dr. R. D. Kleeman's Atomic and Molecular Forces of Chemical and Physical Interaction in Liquids and Gases and their Effects, 1079; Mr. D. Brunt's The Combination of Observations, 1079; Mr. F. P. Ramsey's Foundations of Mathematics, 1209; Dr. H. Jeffreys's Scientific Inference, 1121: Cartesian Tensors, 1211.

Boron fluoride, a search for the band spectra of, 501.

Bowen (E. G.) and Jones (W. Morris) on an X-ray investigation of the bismuth-antimony alloys, 1029.

Bradley (R. S.) on the cohesive force between solid surfaces and the surface energy of solids, 853.

Bramhall (E. H.) on Langmuir probe measurements in the normal copper arc, 682.

British carboniferous reticulate spiriferidæ, 533.

Brown (G. B.) on sensitive flames, 161.

Brown (R. C.) on the surface tensions of mixtures of normal propyl alcohol and benzene, 578.

Burch (F. P.) on potential dividers for cathode ray oscillographs, 760.

Cæsium vapour, an attempt to detect high photoelectric absorption at double the series limit, 300.

Capacity, effective, of the quadrant electrometer, 650.

Characteristic curves of the aluminium rectifying cell, 76.

Charge carried by atoms of radium D emitted by $\alpha$-ray recoil from a source of radium C on a metallic surface and its relations with the surface forces, 1.

Childs (Dr. E. C.) on the collisional friction on electrons in gases, 873.

Circular disk, circular flow in, 674.

Clarke (T. E.) on the effect of surface changes on the photoelectric emission of silver and gold, 624.

Classical energy and the interpretation of Schrödinger's $\psi$ function, 112.

Coagulation of homogeneous aerosols, the theory of, 523.

Cohesive force between solid surfaces and the surface energy of solids, 853.

Collisional friction on electrons moving in gases, 873.

Continuous atomic matrix, 664.
Copper-cadmium series of alloys, the Hall effect and other physical properties of, 201.
Crowther (Prof. J. A.) and Orton (L. H. H.) on the absorption of X-rays in gases and vapours, 505.
Crystal parameters of some of the elements, precision measurements of, 1020.
Current flow in a circular disk, 674.
Davies (W. G.) and Lattey (R. T.) on the effect of electrolytes on the dielectric constant of water, 444.
Davis (A. W.) on the electro capillary effect of capillary-active organic molecules, 1188.
Dean (W. R.), note on the slow motion of fluid, 585.
Deflexion of beams and struts, the more accurate calculation of, 310.
Degenerate gas, transport phenomena in, 361.
Desai (K. V.) and Prasad (Dr. M.) on an X-ray investigation of the crystals of *o*-azotoluene, 600.
Dielectric constant and power factor of some solid dielectrics at radio frequencies, 986.
—— —— of nitrogen over a wide temperature range at high pressures, 1192.
—— —— of water, effect of electrolytes on, 444.
Differential equations, a simple method for the numerical solution of, 1006.
Diffraction patterns due to small circular apertures, 154.
Discharges maintained by electrical oscillations in solenoids, 1179.
Discrimination of differences in brightness, effect of spatial induction on, 975.
Dufton (A. F.) on the graphic computation of solar altitude, 1128.
Elastic extension of metal wires under longitudinal stress, 49.
—— ring acted upon by equal and equiangular radial forces, 705.
Electrical oscillations of very short wave-length, 733, 734.
Electrocapillary effect of capillary-active organic molecules, 1188.
Electrodeless discharges, 745; measurement of current in, by means of frequency variations, 953; theory of electromagnetic and electrostatic induction in, 964.
Electrolytes, effect on the dielectric constant of water, 444.
Electromagnetic waves and pulses, 1049.
Electronic energy levels of the elements and the sizes and sfates of atoms in metallic crystals, 196.
El-Sherbini (M. A.) on the third order terms in the theory of the Stark effect, 24.
Entropy, strain, and the Pauli exclusion principle, 1131.
Equations of motion of a viscous fluid, 615.
—— of state, the theory of, 604.
Ethyl alcohol, standard electrode potentials in, 425.
Evans (Prof. E. J.) and Lewis (W. J.) on the magneto-optical dispersion of acetic anhydride, normal butyric acid, and normal ethyl butyrate in the ultra-violet region of the spectrum, 265.
—— and Richards (W.) on the Hall effect and other physical properties of the copper-cadmium series of alloys, 201.
Existence of the J-phenomena, 28.
Ferromagnetic substances, the thermoelectric properties of, 393.
Ganguli (A.) on the Raman effect from the standpoint of unimolecular reactions, 306.
Ganguli (R.) and Banerji (D.) on the distribution of space-potential in striated and other forms of high-frequency discharge, 494.
Gatty (O.) and Macfarlane (A.) on activities and the standard state: activity coefficients, 283; electrode potentials, 291.
Geological structure of the Eastern Mendips, 742.
George (Dr. T. N.) on the British carboniferous reticulate spiriferidæ, 533.
Gibbs's adsorption equation for the case of binary mixtures, 806.
Gill (E. W. B.) on electrical oscillations of very short wave-length, 734.

Granitic magma and limestone at Palabora, Transvaal, 872.
Graphic computation of solar altitude, 1128.
Hall effect and other physical properties of the copper-cadmium series of alloys, 201.
Hanstock (R. F.) on the effect of mechanical working on the state of a solid surface, 81.
Harmonic continuous calculating machine, 413.
Hartley (Sir H.) and Macfarlane (A.) on standard electrode potentials in ethyl alcohol, 425.
Harwood (Dr. H. F.) on the petrology of the volcanic fields east and south-east of Ruwenzori, Uganda, 327.
Heat conduction and convection from a low hot vertical plate, 888.
—— flow through a granulated material, application of the principle of similitude, 1186.
Heusler alloys, magneto-resistance and magneto-caloric effects in, 233.
Hicks (Prof. W. M.) on the *nhν* emission in xenon and thallium iii, 329.
Higab (Dr. M. A.) on periodic orbits in a field of force defined by a certain potential, 710.
Hoare (F. E.) on a determination of the Stefan-Boltzmann radiation constant, using a Callendar radio-balance, 380.
Holmes (Prof. A.) on the petrology of the volcanic fields east and south-east of Ruwenzori, Uganda, 327.
Hume (Dr. W. F.) on the Pre-Cambrian rocks of Egypt: their nature, classification, and correlation, 534.
Hume-Rothery (Dr. W.) on the electronic energy levels of the elements and the sizes and electronic states of atoms in metallic crystals, 196.
Hume-Rothery relationship between the ionization potentials of the elements and their atomic number, 1163.
Iball (J.) and Owen (Prof. E. A.) on precision measurements of the crystal parameters of some of the elements, 1020.
Integrals, values of, by operational methods, 1144.
Interaction of radiation and the electron, 69.
Ionization potentials of the elements and their atomic number, Hume-Rothery relationship, 1163.
Iron-chromium alloys, X-ray examination of, 419.
Iron, magneto-resistance and magneto-caloric effects in, 233.
Jackson (Dr. G. C. A.) on the geology of the N'Changa district, Northern Rhodesia, 1080.
Jackson (W.) on the transient response of the triode valve equivalent network, 143.
Jeffcott (Dr. H. H.) on the more accurate calculation of the deflexion of beams and struts, 310.
Johnson (P.) on the light intensities of neon discharges, 487.
Johnson (Dr. R. C.) and Tawde (N. R.), a search for the band spectra of boron fluoride, 501.
Jones (W. M.) and Bowen (E. G.) on an X-ray investigation of the bismuth-antimony alloys, 1029.
J-phenomena, the existence of, 28.
Katwe crater-lake, Uganda, 326.
Kimball (W. S.) and Berry (G.) on entropy, strain, and the Pauli exclusion principle, 1131.
—— and King (W. J.) on the theory of heat conduction and convection from a low hot vertical plate, 888.
Kinetics of a catalysed isomeric change in solution, 225.
King (W. J.) and Kimball (W. S.) on the theory of heat conduction and convection from a low hot vertical plate, 888.
Kleeman (Dr. R. D.) on the interaction of radiation and the electron, 69.
Kothari (D. S.) on the transport phenomena in a degenerate gas, 361.
Kuno (Prof. J.) on the application of the law of photo-elastic extinction to some problems, 810.
Kunz (Prof. J.) on the theory of electromagnetic and electrostatic induction in electrodeless discharges, 964.

Laminar motion of viscous fluids, the stability of, 714.
Langmuir probe measurements in the normal copper arc, 682.
Larard (C. E.) on special examples of the elastic ring acted upon by equal and equiangular radial forces, 705.
Laszlo (H. de) on the photoelectric properties of films of beryllium, aluminium, magnesium, and thallium, 1171.
Lattey (R. T.) and Davies (W. G.) on the effect of electrolytes on the dielectric constant of water, 444.
Lattice distortion and hardness of heat-treated tungsten magnet steels, 355.
Lavas of Mauritius, 872.
Lewis (W. J.) and Evans (Prof. E. J.) on the magneto-optical dispersion of acetic anhydride, normal butyric acid, and normal ethyl butyrate in the ultra-violet region of the spectrum, 265.
Light absorption, the Raman effect, and the motions of electrons in gases, 993.
—— intensities of neon discharges, 487.
Lowenstern (Miss E. R.) on the stabilizing effect of imposed oscillations of high frequency on a dynamical system, 458.
Lowery (Dr. H.) and Moore (R. L.) on certain variations in the optical constants of copper, 938.
Lowry (H. V.) on the operational calculus: definition of an operational representation of a function and some properties of the operator derived from this definition, 1033; values of certain integrals and the relationships between various polynomials and series obtained by operational methods, 1144.
Macfarlane (A.) and Gatty (O.) on activities and the standard state: activity coefficients, 283; electrode potentials, 291.
—— and Hartley (Sir H.) on standard electrode potentials in ethyl alcohol, 425.
McGee (J. D.) on the charge carried by $\alpha$-ray recoil atoms of radium D emitted from a source of radium C deposited on a metallic surface and its relations with the surface forces, 1.
McHenry (Dr. J. J.) on the effective capacity of the quadrant electrometer, 650.
McLachlan (Dr. N. W.), experiments on moving-coil reproducers and on flexible disks, 115.
McLennan (Prof. J. C.), Allen (J. F.), and Wilhelm (J. O.) on the superconductivity of alloys containing gold and silver, 1196.
Madgwick (Prof. E.) on a theory of the absorption and transmission of water by porous bodies, 632; the determination of the absorption constants of a homogeneous specimen, 641.
Magneto-optical dispersion of acetic anhydride, normal butyric acid, and normal ethyl butyrate in the ultra-violet region of the spectrum, 265.
—— -resistance and magneto-caloric effects in iron and Heusler alloys, 233.
Manley (Dr. J. J.) on the variations in the refractive index of benzene during intensive drying, 249.
Mechanical working, effect on the state of a solid surface, 81.
Meksyn (Dr. D.) on the wave equations of an electron in a real form, 834.
Mendips, Eastern, geological structure of, 742.
Metal wires under longitudinal stress, elastic extension of, 49.
Methane, liquid and gaseous, Raman spectra in, 455.
Michels (A. and C.) on the dielectric constant of nitrogen over a wide temperature range at high pressures, 1192.
Mitra (Prof. S. K.) and Sil (B. C.) on the variation of resistance of thermionic valves at high frequencies, 1081.
Moore (R. L.) and Lowery (Dr. H.) on certain variations in the optical constants of copper, 938.
Morgans (W. R.) on a continuous atomic matrix, 664.
Motions of electrons in gases, light absorption, and the Raman effect, 993.

Moving-coil reproducers and flexible disks, experiments on, 115.
N'Changa district, Northern Rhodesia, geology of, 1080.
Neon discharges, light intensities of, 487.
*nhν* emission in xenon and thallium iii, 329.
Niessen (Dr. K. F.) and van der Pol (Dr. B.) on symbolic calculus, 537.
Numerical solution of differential equations, 1006.
Oberbeck's vortices, 865.
Omara (Dr. M. A.) on the relativistic precession of periodic orbits in central force fields, 722.
Operational calculus, 1033, 1144.
Optical constants of copper, variations in the, 938.
Orton (Lt. H. H.) and Crowther (Prof. J. A.) on the absorption of X-rays in gases and vapours, 505.
Owen (Prof. E. A.) and Iball (J.) on precision measurements of the crystal parameters of some of the elements, 1020.
Paris (Dr. E. T.) on the sound generated by a rotating airscrew, 99.
Patterson (H. S.) on the theory of coagulation of homogeneous aerosols, 523; on the theory of electrification of aerosols: a correction, 736.
Pauli exclusion principle, entropy, and strain, 1131.
Pendulums immersed in a viscous fluid, on the damping of, 1099.
Periodic orbits in central force fields, relativistic precession of, 722.
—— —— in a field of force defined by a certain potential, 710.
Petrology of the volcanic fields east and south-east of Ruwenzori, Uganda, 327.
Philpot (J. St. L.) on Stern's theory of the electrolytic double layer, 775.
——, Ross-Kane (N. L.), and Wolfenden (J. H.) on the sodium amalgam electrode in dilute solutions, 795.
Photo-elastic extinction, the law of, 810.
Photoelectric absorption in cæsium vapour at double the series limit, 300.
Photoelectric emission of silver and gold, the effect of surface changes on, 624.
—— properties of films of beryllium, aluminium, magnesium, and thallium, 1171.
Piezo-electricity, an application to microscopy, 297.
Polynomials and series, relationships between, by operational methods, 1144.
Potential dividers for cathode ray oscillographs, 760.
Potter (Dr. H. H.) on the magneto-resistance and magneto-caloric effects in iron and Heusler alloys, 233.
Prasad (Dr. M.) and Desai (K. V.) on an X-ray investigation of the crystals of *o*-azotoluene, 600.
Pre-Cambrian rocks of Egypt: their nature, classification, and correlation, 534.
Prescott (Dr. J.) on the equations of motion of a viscous fluid, 615.
Press (Prof. A.) on classical energy and the interpretation of Schroedinger's $\psi$-function, 112.
Preston (G. D.) on an X-ray examination of iron-chromium alloys, 419.
Quadrant electrometer, the effective capacity of, 650.
Quartz, anomalous variation of the electrical conductivity of, 907.
Radiation and the electron, interaction of, 69.
Raman effect from the standpoint of unimolecular reactions, 306.
—— ——, light absorption, and the motions of electrons in gases, 993.
—— spectra in liquid and gaseous methane, 455.
Rates of condensation and evaporation in intensively dried systems, 1014.
Refractive index of benzene, variations during intensive drying, 249.
Relativistic precession of periodic orbits in central force fields, 722.
Resistance of thermionic valves at high frequency, on the variation of, 1081.

Richards (W.) and Evans (Prof. E. J.) on the Hall effect and other physical properties of the copper-cadmium series of alloys, 201.
Richey (J. E.) on the tertiary ring complex of Slieve Gullion, 741.
Rideal (Prof. E. K.) and Schofield (Dr. R. K.) on Gibbs's adsorption equation for the case of binary mixtures, 806.
Robertson (Dr. D.) on the vibrations of revolving shafts, 862.
Robertson (Dr. J. M.) on a simple harmonic continuous calculating machine, 413.
Rosenblatt (A.) on the stability of laminar motion of viscous fluids, 714.
Ross-Kane (N. L.), Philpot (J. St. L.), and Wolfenden (J. H.) on the sodium amalgam electrode in dilute solutions, 795.
Rostagni (A.) on electrical oscillations of very short wave-length, 733.
Ruhuhu coalfields, Tanganyika territory, geology of, 743.
—— river depression, fossil plants from the Karroo beds, 743.
Saunders (O. A.) on the use of the principle of similitude to the problem of heatflow through a granulated material, 1186.
Schroedinger's $\psi$ function, classical energy and the interpretation of, 112.
Schofield (Dr. R. K.) and Rideal (Prof. E. K.) on Gibbs's adsorption equation for the case of binary mixtures, 806.
Sensitive flames, 161.
Shand (Prof. S. J.) on the reaction between granitic magma and limestone at Palabora, Transvaal, 872; the lavas of Mauritius, 872.
Shearer (J. F.) on the effect of spatial induction on the discrimination of differences in brightness, 975.
Shimizu (S.) on the anomalous variation of the electrical conductivity of quartz with temperature, 907.
Sil (B. C.) and Mitra (Prof. S. K.) on the variation of resistance of thermionic valves at high frequencies, 1081.
Similitude and the heatflow through a granulated material, 1186.
Slow motion of fluid, 585.
Sodium amalgam electrode in dilute solutions, 795.
Solar altitude, the graphic computation of, 1128.
Solid surface, effect of mechanical working on the state of, 81.
Sound generated by a rotating air-screw, 99.
Stability of laminar motion of viscous fluids, 714.
Stabilizing effect of imposed oscillations of high frequency on a dynamical system, 458.
Standard electrode potentials in ethyl alcohol, 425.
Stark effect, third-order terms in the theory of, 24.
Stefan - Boltzmann radiation constant, a determination of, using a Callendar radio-balance, 380.
Stern's theory of the electrolytic double layer, 775.
Stockley (G. M.) on the geology of the Ruhuhu coalfields, Tanganyika territory, 743.
Striated and other forms of high-frequency discharge, the distribution of space - potential in, 494.
Sumpner (Dr. W. E.) on electromagnetic waves and pulses, 1049.
Superconductivity of alloys containing gold and silver, 1196.
Surface energy of solids and the cohesive force between solid surfaces, 853.
—— tensions of mixtures of normal propyl alcohol and benzene, 578.
Symbolic calculus, 537.
Synge (E. H.) on an application of piezo-electricity to microscopy, 297.
Tawde (N. R.) and Johnson (Dr. R. C.), a search for the band spectra of boron fluoride, 501.
Tertiary ring complex of Slieve Gullion (Ireland), 741.
Theory of equations of state, 604.
—— of the electrification of aerosols: correction, 736.
Thermionic valves, on the variation of resistance of, 1081.

Thermoelectric properties of ferromagnetic substances, 393.
Thomas (Dr. H. H.) on the tertiary ring complex of Slieve Gullion, petrographical notes, 741.
Thomson (Dr. J.), arc, spark, and glow: a note on nomenclature, 824.
Torsional oscillations of an iron wire: effect of variation of period, 934.
Townsend (Prof. J. S.) on electrodeless discharges, 745.
Traill (Dr. R. C.) on the kinetics of a catalysed isomeric change in solution, 225.
Transient phenomena at the breaking of an inductive circuit, 1001.
—— response of the triode valve equivalent network, 143, 735.
Transport phenomena in a degenerate gas, 361.
Triode valve equivalent network, the transient response of, 143.
Tungsten magnet steels, lattice distortion and hardness of, 355.
Tykocinski-Tykociner (Prof. J.) on the measurement of current in electrodeless discharges by means of frequency variations, 953.
Tyler (Dr. E.) on the damping of pendulums immersed in a viscous fluid, 1099.
Tyte (L. C.) on the elastic extension of metal wires under longitudinal stress, 49.
van der Pol (Dr. B.) and Niessen (Dr. K. F.) on symbolic calculus, 537.
Vibrations of revolving shafts, 862.
Viscous fluid, the equations of motion of, 615.
Walton (Prof. J.) on some fossil plants from the Karroo beds in the Ruhuhu river depression, 743.
Wave equations of an electron in a real form, 834.
Wayland (E. J.) on the Katwe crater-lake, Uganda, 326.
Welch (Dr. F. B. A. W.) on the geological structure of the Eastern Mendips, 742.
Wheeler (Dr. T. S.) on the theory of equations of state, 604.
Wilberforce (Prof. L. R.) on diffraction patterns due to small circular apertures, 154.
Wilcken (Dr. J. A.) on the bending of columns of varying cross-section, 845; on the transient phenomena at the breaking of an inductive circuit, 1001.
Wilhelm (J. O.), McLennan (Prof. J. C.), and Allen (J. F.) on the superconductivity of alloys containing gold and silver, 1196.
Wilkins (Dr. F. J.) on the rates of condensation and evaporation in intensively dried systems: the effect of intensive drying on the accommodation coefficient of liquid and solid surfaces for molecules of their own vapours, 1014.
Wolfenden (J. H.), Philpot (J. St. L.), and Ross-Kane (N. L.) on the sodium amalgam electrode in dilute solutions, 795.
Wood (W. A.) on the lattice distortion and hardness of heat-treated tungsten magnet steels, 355.
X-ray absorption in gases and vapours, 505.
—— examination of the iron-chromium alloys, 419; of the bismuth-antimony alloys, 1029.
—— investigation of the crystals of *o*-azotoluene, 600.
Yagoda (H.) on the Hume-Rothery relationship between the ionization potentials of the elements and their atomic number, 1163.
Yarnold (G. D.) on the discharges maintained by electrical oscillations in solenoids, 1179.

END OF THE THIRTEENTH VOLUME.

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